Week 2 (due October 15)

1. Consider scalar QCD (see Week 1, problem 3).

(a) Write the action in the Hamiltonian form, the way we did in class for pure Yang-Mills theory, and identify the Gauss-law constraint. Compute the Poisson bracket of the constraint (smeared by an adjoint-valued test function $\alpha(\vec{x})$) with the gauge field A, the scalar field ϕ, and their conjugate momenta. Show that the constraint generates time-independent gauge transformations for all fields.

(b) Let $F^a(\vec{x})$ denote the Gauss law constraint. Let $\alpha^a(\vec{x})$ and $\beta^a(\vec{x})$ be two adjoint-valued functions. Compute the Poisson bracket of

$$\int d^3x \; F^a \alpha^a$$

and

$$\int d^3x \; F^a \beta^a$$

and show that smeared constraints form an infinite-dimensional Lie algebra, which is isomorphic to the Lie algebra of time-independent gauge transformations.

2. Consider ordinary QCD in the Lorenz gauge. Its Lagrangian is invariant under BRST transformations, whose explicit form is given by Eq. (16.45) in Peskin and Schroeder.

(a) According to the Noether theorem, BRST invariance implies the existence of a conserved current (the BRST current). Derive an explicit expression for the BRST current.

(b) Let J^μ be the BRST current. We define the BRST charge Q to be

$$Q = \int d^3x \; J^0(\vec{x}).$$

Show by an explicit computation that the Poisson bracket $\{Q, Q\}$ vanishes. (Note that since BRST symmetry is odd, Q is also odd. For odd quantities, the Poisson bracket is symmetric, rather than anti-symmetric, so the vanishing of $\{Q, Q\}$ is not automatic. Upon quantization, this equation becomes $\hat{Q}^2 = 0$, which is the key equation of the BRST formalism.)