Week 4 (due Feb. 4)

1. A little bit of topology.

Let f and g be maps from a topological space X to a topological space Y. For simplicity, you may think of X and Y as manifolds, i.e. spaces which locally look like a Euclidean space. Then in local coordinates f and g are just vector-valued functions depending on local coordinates on X. A homotopy between f and g is a family of maps F_t from X to Y, depending continuously on a parameter $t \in [0,1]$, such that $F_0 = f$ and $F_1 = g$. If such F_t exists, one says that f and g are homotopic and writes $f \sim g$. Now consider a space X and some point $x_0 \in X$. There are two special maps from X to X in this situation: the identity map id_X which sends any point $x \in X$ to itself, and the constant map c_{x_0} which sends any point $x \in X$ to x_0. The space X is called contractible if id_X is homotopic to c.

(a) Show that the Euclidean space $\mathbb{R}^n$ and a unit n-dimensional ball in $\mathbb{R}^n$ are both contractible. (The choice of x_0 does not matter here, so you may choose it to be the origin in both cases.)

The n-th homotopy group of X is defined as the set of all continuous maps from a unit n-dimensional sphere S^n to X, modulo homotopy (i.e. homotopic maps are regarded as identical). It is assumed here that we have marked a point p on S^n, and only consider maps which send p to x_0. In particular, we only consider homotopies F_t such that p is mapped to x_0 for all $t \in [0,1]$. The n-th homotopy group of X is denoted $\pi_n(X)$. Note that a map from S^n to X is the same thing as a map from a unit n-dimensional cube I^n to X which sends the boundary of the cube ∂I^n to x_0. This point of view is convenient for defining the group product and inverse on $\pi_n(X)$, as explained in class. Namely, if $f(t_1, \ldots, t_n)$ and $g(t_1, \ldots, t_n)$ are maps from I^n to X as above, then their composition $f \star g$ is defined as the following map:

$$(f \star g)(t_1, \ldots, t_n) = \begin{cases} f\left(2t_1, t_2, \ldots, t_n\right) & \text{if } t_1 \leq 1/2, \\ g\left(2t_1 - 1, t_2, \ldots, t_n\right) & \text{if } t_1 > 1/2. \end{cases}$$

(b) Show that although $(f \star g) \star h \neq f \star (g \star h)$ in general, we still have

$$(f \star g) \star h \sim f \star (g \star h).$$

Infer from this that the product operation in $\pi_n(X)$ is associative.

(c) The inverse of $f : S^n \to X$ is defined as follows:

$$f^{-1}(t_1, \ldots, t_n) = f(1 - t_1, t_2, \ldots, t_n).$$

Show that $f \star f^{-1}$ is homotopic to the constant map which sends all of S^n to x_0.

(d) Show that if X is contractible, then all its homotopy groups are trivial. (In other words, show that for a contractible X any map from S^n to X is homotopic to the constant map.)

Note: for basic notions of topology, including homotopy groups, see a book by V.A. Vassiliev, "Introduction to Topology," or the book by Nakahara mentioned previously.

2. (a) Let $U(x)$ be an $SU(N)$-valued function on $\mathbb{R}^n$. Show that the n-form $\mathrm{Tr}(U^{-1}dU)^n$ vanishes for even n.

(b) Show that the result of part (a) implies the conservation of the "topological" current constructed from the chiral field $U(x)$ in four dimensions:

$$J^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(U^{-1}\partial_\nu U U^{-1}\partial_\rho U U^{-1}\partial_\sigma U).$$

(The current is called topological because it is conserved even without using the equations of motion).