Week 5 (due Feb. 11)

1. (a) Consider a gauge theory in $d = 3$ with gauge group G (which is a compact simple Lie group). For which G is a global gauge anomaly, similar to Witten's anomaly in $d = 4$, possible? (Hint: you will need a table of homotopy groups for group manifolds, which can be found in Appendix B of Chapter 23 of Weinberg.)

(b) Show that the integral of the Chern-Simons 3-form has the property that it is invariant under all "small" gauge transformations, but not necessarily under "large" gauge transformations (in $d = 3$). Thus a global anomaly in $d = 3$ can show up (and does show up) as a Chern-Simons term in the effective action obtained by integrating out the matter fields.

2. (a) Infinitesimal conformal transformations act on coordinates as follows:

$$[-iK^\mu, x^\nu] = -g^{\mu\nu}x^2 + 2x^\mu x^\nu.$$

The Lie algebra of the conformal group includes the following generators: momenta, angular momenta, Lorenz generators, dilatation, and infinitesimal conformal transformations. Given the knowledge of how these generators act on coordinates, one can realize them by differential operators on functions on space-time. Write down these differential operators and use them to compute the commutation relations of the Lie algebra of the conformal group.

(b) Consider a theory of a free massless scalar field in n dimensions. Compute the dimension of the scalar field which makes the theory scale invariant. Show that the theory is also conformally-invariant, and determine the transformation properties of the scalar field under infinitesimal conformal transformations.

3. In this problem we consider a 2d analogue of the chiral Lagrangian. This theory is called the Wess-Zumino-Witten model and has a number of remarkable properties (not shared by its higher-dimensional cousins), such as conformal invariance and complete integrability on the quantum level. The WZW model describes a scalar field g which takes values in a compact simple Lie group G. The action is

$$S = \int d^2x \ \mathrm{Tr}(\partial_\mu g \partial^\mu g^{-1}) + \frac{k}{24\pi^2} \cdot \int_{B^3} \mathrm{Tr}(g^{-1}dg)^3.$$

Here the last term is the WZW term, which written as an integral of a 3-form over a 3-dimensional ball whose boundary is our space-time. Derive the

equations of motion for the WZW model. (Hint: when varying the WZW term, it is best to refrain from introducing indices until the very end, and use only facts like Stockes theorem, the cyclic property of the trace, and the fact that the wedge product of forms is supercommutative.)