Week 6 (due Feb. 18)

1. We define a 4-vector σ_μ, whose elements are 2×2 matrices, as follows:

$$\sigma_0 = 1, \quad \sigma_i = -i\tau_i,$$

where τ_i are Pauli matrices. 't Hooft tensors $\eta_{a\mu\nu}$ and $\bar{\eta}_{a\mu\nu}$ are defined by the following equations:

$$\sigma_\mu \sigma_\nu^\dagger = \delta_{\mu\nu} + i\eta_{a\mu\nu}\tau_a, \quad \sigma_\mu^\dagger \sigma_\nu = \delta_{\mu\nu} - i\bar{\eta}_{a\mu\nu}\tau_a.$$

(a) Show that $\eta_{a\mu\nu}$ is self-dual and $\bar{\eta}_{a\mu\nu}$ is anti-self-dual for $a = 1, 2, 3$.

(b) Let $g(x)$ be an $SU(2)$ gauge transformation on $\mathbb{R}^4 \backslash \{0\}$ defined as follows:

$$g(x) = \frac{x_\mu \sigma_\mu}{r}, \quad r = |x|.$$

Consider an $SU(2)$ gauge field of the form

$$A_\mu = f(r) g(x) \partial_\mu g(x)^{-1}.$$

Compute the gauge-field strength $F_{\mu\nu}$ for this gauge field and write the answer in terms of 't Hooft tensors.

(c) Let us assume that $f(+\infty) = 1$ and $f(0) = 0$. Show that with this assumption F is self-dual if and only if $f(r)$ has the form

$$f(r) = \frac{r^2}{r^2 + \rho^2},$$

where ρ is an arbitrary number. Compute $F_{\mu\nu}$ for such a gauge field and show that it decays like $1/r^4$.