Week 7 (due Feb. 25)

1. Consider a rigid rod of length l and with vanishingly small mass, attached at one end so that it is free to swing in a vertical plane. A point-like electric charge q of mass M is attached to the other end of the rod. The resulting system is a simple pendulum. Imagine further that there is an infinitely thin solenoid carrying magnetic flux Φ. The solenoid is orthogonal to the oscillation plane and its position coincides with the axis of rotation of the rod. (The set-up is similar to that needed to observe the Aharonov-Bohm effect.) Let g be the free-fall acceleration.

(a) Write down a Lagrangian describing this system, both in the usual and Euclidean signature. Show that classical equations of motion do not depend on Φ.

(b) Write down the quantum-mechanical Hamiltonian for this system. Do the eigenvalues of the Hamiltonian depend on Φ?

(b) This system has a single classical ground state ("vacuum"), but the space of paths from the vacuum to itself has infinitely many disconnected pieces labeled by an integer. This integer measures the net number of turns the pendulum makes. Let us call this integer the winding number. By definition, the instanton solution is the path from the vacuum to itself which has winding number one, minimizes the Euclidean action, and takes infinite time. Find the instanton solution and compute its classical action.

(c) Show that the dependence of the ground-state energy on Φ is non-perturbative in $\hbar$, i.e. to any finite order in $\hbar$ it is Φ-independent. Estimate the dependence of the ground-state energy on Φ using the instanton methods, as described in class. You do not need to evaluate the resulting determinants.

2. Consider scalar QED in $d = 2$ (one-dimensional space). Imagine that we impose periodic boundary conditions in the spatial direction.

(a) Show that the space of classical vacua is classified by an integer. What is the relevant homotopy group in this case?

(b) Since there are multiple vacua, there is tunneling, described by instantons. What is the $d = 2$ analogue of the topological charge ν? Do you expect to have instantons of all sizes, or only of some particular size?

(c) Show that the θ-vacuum corresponds to having a constant electric field in the spatial direction and estimate the dependence of the expectation value of the electric field on θ. (Hint: to evaluate the expectation value of the electric field in a θ-vacuum, note that it is proportional to the logarithmic

derivative of the partition function with respect to θ. Then evaluate the partition function using instanton methods and compute its derivative.)