Week 9 (due March 10)

1. Consider the Georgi-Glashow model, i.e. $SU(2)$ gauge theory in $3+1$ dimensions with a Higgs field $\phi^a, a = 1, 2, 3$ in the adjoint representation. In the vacuum we must have $\phi^a \phi^a = v^2$, so at infinity ϕ^a takes values in a 2-sphere in isospin space. Hence to any static configuration of the Higgs field which tends to a vacuum state at spatial infinity one can assign an integral topological charge, measuring the degree of the mapping from the sphere at spatial infinity S^2_∞ to the sphere in isospin space. As explained in class, one can always find two $SU(2)$-valued transformations U_N and U_S defined on on the complements of the south and north pole on S^2_∞, respectively, so that

$$U_N^{-1}(x)\phi(x)U_N(x) = v \cdot \sigma^3, \; U_S^{-1}(x)\phi(x)U_S(x) = v \cdot \sigma^3.$$

Here we denoted by ϕ the matrix-valued field $\phi^a \sigma^a$, and σ^a are Pauli matrices.

Consider the special case of a "hedghog" Higgs field, i.e.

$$\phi^a = v \frac{x^a}{|x|}.$$

Construct the gauge transformations U_N and U_S, compute $\Omega = U_N^{-1} U_S$ (the gauge transformation which "connects" the two patches on S^2_∞), and show that it is a $U(1)$-valued transformation which has unit winding number around the equator of S^2_∞. (As explained in class, this means that the hedgehog configuration has unit magnetic charge).

2. (ab) Consider the Georgi-Glashow model in the limit $\lambda \to 0$, i.e. drop the potential term for the Higgs field. The energy functional is

$$E(A, \phi) = \int d^3x \left(\frac{1}{4} \left(F^a_{ij} \right)^2 + \frac{1}{2} (D_i \phi^a)^2 \right)$$

Let us consider field configurations for which $\phi^a \phi^a$ tends to a fixed constant v at infinity. Show that for such field configuration one has an inequality:

$$E(\phi, A) \geq 4\pi v Q_m.$$

Here Q_m is the magnetic charge:

$$Q_m = \frac{1}{4\pi v} \int_{S^2_\infty} B^a_i \phi^a d\Sigma^i,$$

and $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$. Show that the inequality becomes an equality iff ϕ and A satisfy the Bogomolny equations:

$$(D_i\phi)^a = \pm\frac{1}{2}\epsilon_{ijk}F_{jk}^a.$$

(c) Find a solution of the Bogomolny equation with $Q_m = 1$ (the one-monopole solution). (Hint: use the spherically symmetric hedgehog ansatz written down in class).