Week 5 (due May 7)

Reading: Srednicky, sections 71, 72. See also Peskin-Schroeder for a better explanation of the Faddeev-Popov gauge-fixing procedure.

1. (a) Derive the equations of motion following from the Yang-Mills action

$$S = -\frac{1}{2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}.$$

(b) A gauge field A_μ is called flat if $F_{\mu\nu} = 0$. Show that if a gauge field A is gauge transformation of the zero gauge field, i.e. if it has the form

$$A_\mu(x) = iU(x)\partial_\mu U^{-1}(x)$$

for some gauge transformation U, then it is flat. The converse is actually also true (on $\mathbb{R}^4$), but it is harder to prove. Show that a flat gauge field solves the Yang-Mills equations of motion derived in part (a). (Such solutions of course are rather trivial, in the sense that they are gauge-equivalent to the zero solution).

2. (a) Prove the Bianchi identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0.$$

Note that the covariant derivative here is the one appropriate for the adjoint representation.

(b) Consider the equation

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^{\rho\alpha}\eta^{\sigma\beta}F_{\alpha\beta}.$$

This equation is called the instanton equation, and its solutions are called instantons. Using the Bianchi identity from part (a) show that any solution of the instanton equation solves the Yang-Mills equations of motion (the converse is not necessarily true). Show further that if the metric is the Minkowski metric, then the instanton equations are equivalent to $F_{\mu\nu} = 0$, and so all instantons are gauge-equivalent to zero. Show that if the metric is the Euclidean metric (i.e. the identity matrix), then the instanton equations do not imply $F_{\mu\nu} = 0$, and thus there may be nontrivial solutions. (In fact, there are many solutions of the instanton equation on $\mathbb{R}^4$).

3. (a) Show that the expression

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma}$$

is a total derivative:

$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma} = \partial_\mu K^\mu.$$

Hint: use the following ansatz for K^μ:

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(a\, A_\nu \partial_\rho A_\sigma + b\, A_\nu A_\rho A_\sigma),$$

and tune the numbers a and b to get the desired identity. Note that the last term in K^μ does not vanish because A_ρ and A_σ are matrices which need not commute.

(b) Having found a and b and K^μ, let us set $\mu = 0$. Clearly K^0 is a function of the spatial components of A only. Thus function is called the Chern-Simons density. We may regard it as a scalar function of the 3d gauge field with components A_1, A_2, A_3. Show that under an infinitesimal gauge transformation

$$\delta A_\mu = -D_\mu \epsilon, \quad \mu = 1, 2, 3, \quad \epsilon = \epsilon(x^1, x^2, x^3),$$

the Chern-Simons density changes by a total derivative. Therefore the integral of the Chern-Simons density is gauge-invariant and can be used as a candidate for the action of a 3d gauge theory. This action is called the Chern-Simons action and is special to 3d. (In contrast, the Yang-Mills action makes sense in all dimensions).

(c) Derive the equations of motion arising from varying the Chern-Simons action. Show that they are equivalent to

$$F_{\mu\nu} = 0.$$

If the space-time is $\mathbb{R}^3$, this equation implies that A_μ is gauge-equivalent to zero. Thus Chern-Simons gauge theory is rather trivial (has no propagating degrees of freedom), unlike Yang-Mills theory.