Week 8 (due March 5)

1. Consider a fermionic spinor field in two space-time dimensions. In two dimensions one can take $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, where σ^k is a Pauli matrix. Therefore the Dirac spinor has only two complex components. Let the Lagrangian be

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{g}{4}(\bar{\psi}\psi)^2.$$

(a) Derive the Feynman rules for this theory, with all Z factors and μ-dependence included.

(b) Classify divergent Green's functions.

(c) Compute the divergent parts of Z_ψ (the renormalization factor for ψ) and Z_g (the renormalization factor for the interaction term) at one loop using dimensional regularization. Don't forget that a fermionic loop gives an extra minus sign!

(d) Compute the beta-function for g at one-loop order. You should get that the theory is asymptotically free, i.e. the coupling goes to zero in the limit $\mu \to \infty$.

2. Consider a theory of a vector field in three-dimensional space-time with a Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + k\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho.$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The second term in the Lagrangian is called the Chern-Simons term, and the theory is called Chern-Simons-Maxwell theory.

(a) Derive the equations of motion for the field A. Show that they imply that each component of the tensor $F_{\mu\nu}$ satisfies a massive Klein-Gordon equation for some mass m. Express m in terms of k.

(bc) In three dimensions the analog of the electric field $E_i = -\partial_0 A_i + \partial_i A_0$ is a vector, but the analog of the magnetic field $B = \epsilon_{ij}\partial_i A_j$ is a scalar. As in four dimensions, A_0 should be thought of as a Lagrange multiplier whose equation of motion gives a constraint (the analog of the Gauss law) and does not have a conjugate momentum, while the fields A_i, $i = 1, 2$, do. What is the analog of the Gauss law constraint $\partial_i E_i = 0$ in this theory ? Find the momenta conjugate to A_i. Write the Hamiltonian in terms of B and the momenta conjugate to A_i. Write the action in Hamiltonian form (i.e. in a form where the momenta and A_i are regarded as independent and the action is linear in time derivatives).

(d) Solve the equations of motion for A in the Coulomb gauge $\nabla \cdot A = 0$ and write down the solution in terms of creation-annihilation operators. Show that the theory describes a massive particle of mass m with a single polarization state.

(e) Now add a source term $J^\mu A_\mu$ to the Lagrangian, where J^μ is a current density. Consider the case when $J^0 = Q\delta^3(x)$, $J^i = 0$, $i = 1, 2$. That is, there is a charge Q at rest localized at the origin. Find the fields E_i and B created by such a source (i.e. find the unique time-independent solution of equations of motion which decays at infinity).