Week 5 (due Nov. 2)

1. The transition amplitude in nonrelativistic Quantum Mechanics is defined by

\[ K(q', q; T) = \langle q'|e^{-iHT}|q \rangle. \]

Here \( H \) is the usual Hamiltonian, i.e.

\[ H = \frac{\hat{p}^2}{2m} + V(q). \]

(a) Compute \( K(q', q; T) \) for the free particle (\( V = 0 \)) by inserting identity operators in the form

\[ 1 = \int dp |p\rangle \langle p|, \]

and then evaluating the resulting matrix elements and integrals over \( p \).

(b) On the other hand, one can consider the second-quantized version of the same system and the corresponding 2-point Green’s function

\[ G(q', q; T) = \langle 0|\Psi(T,q')\Psi^\dagger(0,q)|0 \rangle \]

Show that for any potential \( V(q) \) one has \( G(q', q; T) = K(q', q; T) \). Verify this in the special case \( V = 0 \) by directly evaluating \( G(q', q; T) \) using the known Fourier-expansion of \( \Psi \) and \( \Psi^\dagger \) in terms of creation-annihilation operators and then comparing with the results of part (a).

(c) The path-integral representation for \( K(q', q; T) \) is

\[ K(q', q; T) = \int Dq(t) \exp(iS), \]

where

\[ S = \int dt \left( \frac{m}{2} \dot{q}^2 - V(q) \right). \]

In more detail, the path-integral is defined as the limit

\[ \lim_{N \to \infty} F(\epsilon)^N \int dq_1 \ldots dq_{N-1} \exp \left[ i\epsilon \sum_{i=0}^{N-1} \left( \frac{m}{2} (q_{i+1} - q_i)^2 / \epsilon^2 - V(q_i) \right) \right], \]

where \( \epsilon = T/N, q_0 = q, q_N = q' \), and the function \( F(\epsilon) \) should be chosen so that in the limit \( N \to \infty \) one gets the correct expression for \( K(q', q; T) \).
Determine $F(\epsilon)$ by evaluating the integral in the special case $V = 0$ and comparing with the results of part (a).

(d) Consider now the case $V(q) = -fq$. This corresponds to a particle which is acted upon by a constant force $f$. Find $K(q', q; T)$ by evaluating the path-integral and using the function $F(\epsilon)$ found in part (c).