Week 2 (due Oct. 16)

1. Let ϕ be a free real scalar field. The commutator $\Delta(x) = [\phi(x), \phi(0)]$ is a c-number (i.e. it is proportional to the identity operator in Fock space) and is known as the commutator function for ϕ. Compute the commutator function for the case $m = 0$ (massless free field). Hint: by rotational invariance, you may assume that the spatial part of x is along the x^1 axis. The integral over k_2 and k_3 is easily computed, if we recall that

$$\frac{d^3k}{2\omega_k} = d^4k\, \delta(-k^2)\theta(k^0),$$

where θ is a step-function, i.e. $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$. The remaining integral over k^0 and k^1 is most easily evaluated in the "light-cone coordinates" $k_+ = k^0 - k^1$ and $k_- = k^0 - k^1$.

2. Let ϕ be as in problem 1. Compute the vacuum expectation value

$$< 0|\phi(x)\phi(0)|0 > .$$

Hint: be careful, this is a distribution, not a function. Use the same method as in problem 1.

3. The Hamiltonian for the free complex scalar field of mass m is

$$H = \int d^3x \left(p^\dagger p + \partial_i \phi^\dagger \partial_i \phi + m^2 \phi^\dagger \phi \right).$$

Here $p = \partial_0 \phi^\dagger$ is the momentum conjugate to ϕ and $p^\dagger = \partial_0 \phi$ is the momentum conjugate to $\phi^\dagger$. The nonvanishing equal-time commutators are

$$[p(\vec{x}), \phi(\vec{y})] = -i\delta^3(\vec{x} - \vec{y}), \quad [p^\dagger(\vec{x}), \phi^\dagger(\vec{y})] = -i\delta^3(\vec{x} - \vec{y}).$$

Show that the Heisenberg equations of motion

$$i\partial_0 \phi = [H, \phi], \quad i\partial_0 p = [H, p]$$

are equivalent to the Klein-Gordon equation for ϕ.

4. (a) Consider a field theory with three real scalar fields $\phi^a(x)$, $a = 1, 2, 3$, and a Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi^a(x) \partial^\mu \phi^a(x) - V(\phi^a \phi^a).$$

Here summation over repeating indices a is assumed, and V is an arbitrary function. This Lagrangian is obviously invariant with respect to orthogonal transformations of the fields ϕ^a:

$$\phi^a(x) \mapsto \tilde{\phi}^a(x) = R^a_b \phi^b(x),$$

where R^a_b is a constant orthogonal 3×3 matrix. The rotation group in three dimensional space has dimension three, so we expect to get three conserved currents. Show that infintesimal transformations for $\phi^a(x)$ can be put into the form

$$\delta\phi^a(x) = \epsilon^{abc}\phi^b(x)\beta^c,$$

where $\beta^c, c = 1, 2, 3$ parametrize an infintesimal rotation, and ϵ^{abc} is a completely anti-symmetric tensor uniquely defined by the condition $\epsilon^{123} = 1$. Deduce the conserved currents corresponding to this symmetry.

(b) Let the currents found in part (a) be called $J^{a\mu}, a = 1, 2, 3$. The corresponding charges are

$$Q^a = \int d^3x J^{a0}(x).$$

Compute the commutator of Q^a and Q^b using canonical commutation relations for ϕ^a and their time derivatives. Show that Q^a form a Lie algebra isomorphic to the Lie algebra of the rotation group (i.e. show that they obey the same commutation relations as components of the angular momentum operator in quantum mechanics).