1 Content of the course

"Quantum Field Theory" by M. Srednicki, Part 1.

2 Combining QM and relativity

We are going to keep all axioms of QM:

1. states are vectors (or rather rays) in Hilbert space.
2. observables are Hermitian operators and their values are the spectrum.
3. probability of measuring a particular value a of an observable A in a state Ψ is

$$\frac{||P_a\Psi||^2}{||\Psi||^2},$$

where P_a is a projector to the eigenspace of A corresponding to a.

4. Time evolution of states is governed by the Schrödinger equation

$$i\frac{d\Psi(t)}{dt} = H\Psi(t),$$

where H is the Hamiltonian (energy operator).

5. Symmetries are unitary or anti-unitary operators preserving the Hamiltonian.
6. etc.

For a nonrelativistic particle, we let $\mathcal{H} = L^2(\mathbb{R}^3)$ and let the momentum operator (generator of translations) be $\hat{\mathbf{P}} = -i\hbar\nabla$. Since $E = \mathbf{P}^2/2m$ classically, it is natural to define $H = \hat{\mathbf{P}}^2/2m = -\hbar^2\nabla^2/2m$.

From now on, I will let $\hbar = 1$, so $H = -\nabla^2/2m$.

For a relativistic particle,

$$E = \sqrt{\mathbf{P}^2c^2 + m^2c^4},$$

so can try

$$H = \sqrt{-\nabla^2c^2 + m^2c^4}$$

This expression is problematic: treats time and space asymmetrically and appears nonlocal.

Alternatively, we can "quantize" the squared dispersion relation $E^2 = \mathbf{P}^2c^4 + m^2c^4$ to get the Klein-Gordon equation

$$-\frac{\partial^2}{\partial t^2}\Psi = \left(-c^2\nabla^2 + m^2c^4\right)\Psi.$$

This equations is more reasonable, as it is more symmetric w.r. to exchange of time and space. To see relativistic invariance better, let $x^0 = ct$. From now I will let $c = 1$, so in such units $x^0 = t$. Also, $x_0 = -t$, and $x^i = x_i, i = 1, 2, 3$. Greek indices will run over the set $0, 1, 2, 3$.

Minkowski metric: $g_{\mu\nu} = diag(1, -1, -1, -1) = g^{\mu\nu}$. $x_\mu = g_{\mu\nu} x^\nu$, where we use Einstein's convention (summation over repeated indices). Similarly, $x^\mu = g^{\mu\nu} x_\nu$.

Minkowski interval: $x^2 = x^\mu x_\mu = x^\mu x^\mu g_{\mu\nu} = (x^0)^2 - \sum_i (x^i)^2$.

Lorenz transformations are

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu,$$

where Λ is any real matrix such that $\bar{x}^\mu \bar{x}_\mu = x^\mu x_\mu$.

Now we can check relativistic invariance of the KG equation, i.e. that $\phi(x)$ and $\phi(\bar{x})$ satisfy the same equation.

Let

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \partial^\mu = g^{\mu\nu} \partial_\nu.$$

Then

$$\bar{\partial}^\mu = \Lambda^\mu_\nu \bar{\partial}^\nu.$$

and therefore $\bar{\partial}^2 = \partial^2$. Hence the KG operator is Lorenz-invariant.

Problems:

1. $\int d^3x |\Psi|^2$ is not conserved. Moreover, it has wrong transformation properties under the Lorenz transformation: $|\Psi|^2$ is not a time component of a 4-vector, so we do not expect a continuity equation to hold (and it does not). One can write down something which is a component of a conserved 4-vector:

$$j_\mu = i(\Psi^* \partial_\mu \Psi - \partial_\mu \Psi^* \Psi)$$

satisfies $\partial_\mu j^\mu = 0$, and so

$$\int d^3x j^0$$

is conserved. But j^0 is not positive-definite, so cannot be interpreted as probability density.

2. Negative-energy solutions.

Dirac tried to solve these problems by looking for a first-order equation, but for a multicomponent wavefunction. This solved problem 1, but not problem 2.

Ultimately, the problem is that relativistic QM can be consistently developed only if we do not work in a theory with a fixed number of particles. Hence we need to understand how to describe systems where particle creation and annihilation is allowed.

3 Fock space methods (second quantization)

3.1 Bosons

A single particle has $\mathcal{H}_1 = L^2(\mathbb{R}^3)$ as its Hilbert space. Two particles have $\mathcal{H}_2 = L^2(\mathbb{R}^3 \times \mathbb{R}^3)_{sym} \simeq Sym^2\mathcal{H}_1$. And so on. The Hilbert space without any particles is one-dimensional (the vacuum state). Thus

$$H = \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots = \oplus_{k=0}^{\infty}\mathrm{Sym}^k(\mathcal{H}_1).$$

This is called the bosonic Fock space $\mathcal{F}(\mathcal{H}_1)$ associated to $\mathcal{H}_1$.

The Fock space is always infinite-dimensional, even if $\mathcal{H}_1$ is not. Let us look at the extreme case, $\mathcal{H}_1 \simeq \mathbb{C}$. Then

$$\mathcal{F}(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \ldots.$$

Thus a vector in $\mathcal{F}(\mathbb{C})$ is an infinite sequence of numbers or vector $(a_0, a_1, a_2, \ldots)$ such that $\sum_k |a_k|^2 < \infty$.

It is often convenient to think of such a sequence as Taylor coefficients of an analytic function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Degree is then identified with the particle number. Polynomials form a dense set in this space of functions and correspond to states with a finite number of particles.

Two natural operations on polynomials are z and ∂. They satisfy

$$[\partial, z] = 1.$$

One calls ∂ the annihilation operator a, and calls z the creation operator $a^\dagger$. They are indeed conjugate to each other if we define the scalar product to be

$$||f(z)||^2 = \frac{1}{2\pi}\int d^2z |f(z)|^2 e^{-|z|^2}.$$

Using this scalar product, one can compute $||z^n||^2 = n!$. Thus a normalized n-particle state is

$$|n\rangle = \frac{1}{\sqrt{n!}} z^n = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle.$$

Thus

$$a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle$$

This can serve as a definition of creation and annihilation operators.

The particle number operator can be expressed as $N = z\partial_z = a^\dagger a$. Eigenstates of N are homogeneous polynomials. Polynomials are Fock space states which involve only a finite number of particles.

More generally, suppose $\mathcal{H}_1 \simeq \mathbb{C}^N$. Let us choose a basis $\psi_i, i = 1, \ldots, N$ in $\mathcal{H}_1$ and introduce N variables $z_1, \ldots, z_N$. Then one can identify $\mathrm{Sym}^p(\mathcal{H}_1)$ with the space of polynomials in N variables of total degree p: a state with k_1 particles in the state ψ_1, k_2 particles in the state ψ_2, etc. can be identified with the polynomial

$$z_1^{k_1} \ldots z_N^{k_N}.$$

We define $a_i = \partial_i, a_i^\dagger = z_i$ so that

$$[a_i, a_j^\dagger] = \delta_j^i.$$

The Fock space is then the space of all polynomials in variables $z_1, \ldots, z_N$. If we change the basis in $\mathcal{H}_1$, creation-annihilation operators also change: if $\psi_i' = B_i^j \psi_j$, where B is a unitary matrix, then

$$a_i' = B_j^{*i} a_j$$

The particle number operator is $N = \sum_i z_i \partial_i = \sum_i a_i^\dagger a_i$. Eigenstates of N are homogenous polynomials.

If $\mathcal{H}_1$ is infinite-dimensional, but has a countable basis, we can still think of its Fock space as a completion the space of polynomials in variables $z_1, z_2, \ldots$.

But usual bases on $L^2(\mathbb{R}^3)$ (momentum eigenstates $|\mathbf{p}\rangle$ and coordinate eigenstates $|\mathbf{x}\rangle$ are not like that. Still, one can define analogues of creation and annihilation operators:

$$\Psi(x) = \sum_i a_i \psi_i(x), \quad \Psi^\dagger(x) = \sum_i a_i^\dagger \psi_i^*(x).$$

They satisfy

$$[\Psi(x), \Psi^\dagger(y)] = \delta^3(x-y).$$

All operators in Fock space can be expressed in terms of $\Psi(x)$ and $\Psi^\dagger(x)$.

Examples:

0. The particle number operator $N = \int d^3x \Psi^\dagger(x)\Psi(x)$.

1. One-particle operators. A one-particle operator is an operator of the form

$$\sum_{k=1}^{\infty}\sum_{i=1}^{k} 1\otimes \ldots \otimes 1 \otimes \mathcal{O} \otimes 1 \otimes \ldots \otimes 1 = \sum_{k=0}^{\infty}\sum_{i=1}^{k} \mathcal{O}_i,$$

where $\mathcal{O}$ is an operator on $\mathcal{H}_1$. It can be written as

$$\int d^3x d^3y \Psi^\dagger(x)\langle x|\mathcal{O}|y\rangle \Psi(y).$$

For example, the kinetic energy operator is a one-particle operator with $\mathcal{O} = -\nabla^2/2m$, so the corresponding operator in Fock space is

$$\int d^3x \Psi^\dagger(x)\left(-\nabla^2/2m\right)\Psi(x).$$

The particle number operator is a one-particle operator with $\mathcal{O} = 1$.

2. Two-particle operators. These are operators of the form

$$\sum_{k=1}^{\infty}\sum_{1\le i<j\le k} \mathcal{O}_{ij}$$

where $\mathcal{O}_{ij}$ is an operator on $\mathcal{H}_2$ which acts only on the i-th and j-th particle. The corresponding operator in Fock space is

$$\frac{1}{2}\int d^3x d^3y d^3z d^3t \Psi^\dagger(x)\Psi^\dagger(y)\langle x,y|\mathcal{O}|z,t\rangle \Psi(t)\Psi(z).$$

For example, the potential energy operator is of this form, with $\langle x,y|\mathcal{O}|z,t\rangle = V(x-y)\delta^3(x-z)\delta^3(y-t)$. The corresponding operator in Fock space is

$$\frac{1}{2}\int d^3x d^3y \Psi^\dagger(x)\Psi^\dagger(y)V(x-y)\Psi(y)\Psi(x).$$

How do we formulate dynamics in the Fock space? Since the emphasis is on creation-annihilation operators, it is often convenient to work in the

Heisenberg picture and write EOMs for Ψ and $\Psi^\dagger$, instead of the Schrödinger equation. For free particles, we get

$$\frac{\partial \Psi}{\partial t} = i[H, \Psi] = -\frac{i}{2m}\nabla^2\Psi.$$

This looks like Schrödinger equation, but for a field operator. Hence the name "second quantization". Let us find a solution. Go to momentum space:

$$\Psi(x) = \int d^3p(2\pi)^{-3}b(p)e^{ipx}$$

Then

$$[b(p), b^\dagger(q)] = (2\pi)^3\delta^3(p-q).$$

and

$$H = \int d^3p(2\pi)^{-3}\frac{p^2}{2m}b^\dagger(p)b(p).$$

$$b(p,t) = e^{-iE_pt}b(p,0).$$

Thus

$$\Psi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}b(p,0)e^{-E_pt+i\mathbf{p}\cdot\mathbf{x}}.$$

This completely determines the evolution of all observables.

For an interacting system, get the following equation:

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2m}\nabla^2\Psi + \int d^3y\Psi^\dagger(y)\Psi(y)V(x-y)\Psi(x).$$

This is nonlinear and cannot be regarded as "second-quantized" Schrodinger equation. Its classical analogue is a PDE for an ordinary function $\Psi(t, x)$, which is NOT interpreted as a quantum-mechanical wavefunction.

Remark: the quantum harmonic oscillator corresponds to the Fock space for $\mathcal{H} = 1$. A collection of N harmonic oscillators is equivalent to the bosonic Fock space for $\mathcal{H}_1 = \mathbb{C}^N$.Thus the quantization of a system of harmonic oscillators can be interpreted in terms of free bosonic particles. The energies of 1-particle states are ω_i.

3.2 Fermions

Now consider fermionic particles which obey the Pauli principle. Fermionic wavefunctions are antisymmetric with respect to the exchange of any two particles.

Let us again begin with the case $\mathcal{H}_1 = \mathbb{C}$. Then the Fock space is

$$\mathcal{F}(\mathcal{H}_1) = \mathbb{C} \oplus \mathbb{C}.$$

This is two-dimensional, and there are many ways to think about it. E.g., we can identify it with the states of a spin-1/2 particle. But we will choose a more esoteric viewpoint. Consider a variable θ which has a multiplication rule $\theta^2 = 0$. Then "analytic functions of θ" are linear functions

$$f(\theta) = a + b\theta.$$

The space of such functions can be identified with $\mathcal{F}(\mathcal{H}_1)$: th vacuum state is 1, while the 1-particle state is θ.

Creation-annihilation operators are defined as before: $c^\dagger = \theta$, $c = \partial_\theta$. Note that $c^2 = (c^\dagger)^2 = 0$. It is also easy to check that $cc^\dagger + c^\dagger c = 1$. Note the crucial plus sign. The particle number operator is $N = c^\dagger c$.

We can define the scalar product so that $c^\dagger$ is indeed the adjoint of c. Details of this are left as an exercise.

Now consider N-dimensional $\mathcal{H}_1$. Introduce N variables θ_i which satisfy $\theta_i\theta_j + \theta_j\theta_i = 0$. Consider an analytic function of θ_i. Again, the series terminates in degree N. The total dimension of the space of functions is 2^N. The k-ht term in the expansion is

$$\sum_{i_1 \dots i_k} f^{i_1 \dots i_k} \theta_{i_1} \dots \theta_{i_k}.$$

Here the coefficient functions are totally anti-symmetric, as required by the Fermi statistics. The creation operators are $c_i^\dagger = \theta_i$, the annihilation operators are $c_i = \partial_i$. They satisfy

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}.$$

Note that the fermionic Fock space has a symmetry which replaces the vacuum with the "filled state" $\theta_1 \dots \theta_N$ and exchanges c_i and $c_i^\dagger$. There is nothing analogous in the bosonic case.

The rest proceeds as before. We can choose a countable basis in $L^2(\mathbb{R}^3)$ and define

$$\Psi(x) = \sum_i \psi_i(x) c_i, \quad \Psi^\dagger(x) = \sum_i \psi_i^*(x) c_i^\dagger.$$

They satisfy

$$\Psi(x)\Psi^\dagger(y) + \Psi^\dagger(y)\Psi(x) = \delta^3(x-y).$$

These are called canonical anti-commutation relations. In the noninteracting case, the EOM is linear and solved exactly as in the bosonic case.

4 Classical field theory

There is something special about differential equations which come from "dequantizing" the Heisenberg equations of motion: they come from a variational principle.

4.1 Classical mechanics

Recall classical mechanics. Action:

$$S = \int_0^T dt L(q^i(t), \dot{q}^i(t)).$$

Euler-Lagrange variational principle: $\delta S = 0$ with $q(0)$ and $q(T)$ fixed. Equations of motion:

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right).$$

Alternatively, we can introduce $p_i = \partial L/\partial q^i$, the Hamiltonian

$$H = p\dot{q} - L,$$

and write the action as

$$S = \int dt(p\dot{q} - H(p(t), q(t))).$$

The equation $\delta S = 0$ then gives

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

These are Hamilton equations.

Finally, if we introduce the Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}$$

for any two functions F, G, the Hamilton equations of motion can be written as

$$\dot{q}^i = \{H, q^i\}, \quad \dot{p}_i = \{H, p_i\}.$$

We also have

$$\{p_i, q_j\} = \delta_i^j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0.$$

Under quantization, Poisson bracket becomes i times the commutator.

4.2 Nonrelativistic field theory

Now we want to have a similar formalism where i is replaced with a continuous index $\mathbf{x}$. Instead of $q_i(t)$ will have $\Psi(t, \mathbf{x})$. Action:

$$S = \int dt L(\Psi, \dot{\Psi}).$$

EOM:

$$\frac{\delta L}{\delta \Psi(t, \mathbf{x})} = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\Psi}(t, \mathbf{x})}\right).$$

Here the variational derivative is defined by

$$\delta L = \int d^3x \frac{\delta L}{\delta \Psi(t, \mathbf{x})} \delta \Psi(t, \mathbf{x}).$$

In the free case, it is sufficient to take

$$L = L_0 = \int d^3x \left(i \Psi^* \dot{\Psi} - \frac{1}{2m} \partial_i \Psi^* \partial_i \Psi\right).$$

Note that L is an integral of a local expression, $L = \int d^3x \mathcal{L}$, so

$$S = \int dt d^3x \mathcal{L}(\Psi, \dot{\Psi}).$$

This is very nice, but is not obligatory in a nonrelativistic situation: in an interacting case one finds

$$L = L_0 - \frac{1}{2}\int d^3x d^3y |\Psi(x)|^2 |\Psi(y)|^2 V(\mathbf{x}-\mathbf{y}).$$

This is local in some very special cases. For example, when $V(x) = \delta^3(x)$ ("contact interaction"). In the relativistic case only such interaction are allowed.

Note that this fits better with the second version of the variational principle: $i\Psi^*$ is the "momentum conjugate to Ψ". So one has Poisson brackets

$$\{\Psi^*(\mathbf{x}), \Psi(\mathbf{y})\} = -i\delta^3(\mathbf{x}-\mathbf{y}).$$

The Hamiltonian is then given by the same expression as before, but Ψ's are now ordinary functions, not Fock-space operators.

Quantization now is easy: we get the standard commutation relations for Ψ and Ψ^* and realize them as operators in Fock space.

How do we get fermionic Fock space in this way? There is no good way of doing so. Reason: classical limit makes sense only when a large number of particles are in the same state.

For clarity, consider discrete case. In order for the commutator term to be negligible, need to consider a state where a has a large expectation value (and small variance). Hence $N = a^\dagger a$ will have a large expectation value. This is not possible in the fermionic case.

Formally, we can still consider the same equations, but with Ψ and Ψ^* satisfying anticommutation relations. This means that they are not ordinary functions, but generators of a Grassmann algebra. We will use this trick later.

4.3 Relativistic field theory

Main idea: interpret the KG equation not as an equation for a wavefunction, but an equation for a field operator. That is, let us make relativistic not the one-particle Schrodinger equation, but the Heisenberg equation of motion for the Fock space operator.

To understand it, we need to specify commutation relations for Ψ in such a way, that the KG equation is the Heisenberg equation for some Hamiltonian. We can do this like this: first solve an analogous classical problem, and then quantize everything.

The classical KG equation comes from the action

$$S = \frac{1}{2}\int dtd^3x\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right).$$

This looks more like the first version of the variational principle. The momentum is

$$p(x) = \dot{\phi}(x),$$

and the Hamiltonian is

$$H = \int d^3x\mathcal{H} = \frac{1}{2}\int d^3x\left(p(x)^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

The Poisson brackets are

$$\{p(\mathbf{x}), \phi(\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y}\}.$$

Hence quantization will give

$$[\phi(\mathbf{x}), p(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

This is just like $[q.p] = i$, but with continuous indices.

Reason: the classical system describes the continuum limit of a system of particles connected with springs, and $\phi(x)$ is the continuum limit of the coordinate of a particle.

Classical excitations are waves. What about quantization? Expect that we get a system of free bosonic particles with a relativistic dispersion law. Two reasons: (1) that is what we set out to describe; (2) classical system can be Fourier-analyzed into a collection of harmonic oscillators; each oscillator is equivalent to a Fock space (for a one-dimensional vector space), so the whole thing is equivalent to a Fock space (for an infinite-dimensional 1-particle space), so describes free bosonic particles.