Week 1 (due April 9)

1. (a) The complex symplectic group $Sp(2N, \mathbb{C})$ is a complex subgroup of $GL(2N, \mathbb{C})$ defined by the condition $M^t J M = J$, where J is a block-off-diagonal matrix of the form

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Show that the Lie algebra of $Sp(2N, \mathbb{C})$ can be identified with the space of symmetric $2N \times 2N$ complex matrices and use this fact to compute the (complex) dimension of $Sp(2N, \mathbb{C})$.

(b) The unitary symplectic group $USp(2N)$ is a real subgroup of $GL(2N, \mathbb{C})$ defined as the intersection of $Sp(2N, \mathbb{C})$ and $U(2N, \mathbb{C})$. Compute the real dimension of $USp(2N)$. Show that $USp(2)$ is isomorphic to $SU(2)$.

2. (a) The covariant derivative of a field ψ in the adjoint representation of $G \subset U(N)$ is defined by

$$D_\mu \psi = \partial_\mu \psi - i[A_\mu, \psi].$$

Here we regard ψ is a field valued in $N \times N$ matrices which under gauge transformations transforms as

$$\psi \mapsto U \psi U^{-1}.$$

Show that the covariant derivative transforms as

$$D_\mu \psi \mapsto U (D_\mu \psi) U^{-1}.$$

Now let ϕ be a field which transforms in the anti-fundamental representation of G, i.e. ϕ is a row-vector of length N which transforms as follows:

$$\phi \mapsto \phi U^{-1}.$$

Show that if we define the covariant derivative by

$$D_\mu \phi = \partial_\mu \phi + i \phi A_\mu,$$

then it transforms as follows:

$$D_\mu \phi \mapsto (D_\mu \phi) U^{-1}.$$

(b) Let χ be a field in the rank-2 tensor representation of $G \subset U(N)$, i.e. it is an $N \times N$ complex matrix which transforms as follows:

$$\chi \mapsto U\chi U^t.$$

Write down a formula for the covariant derivative for χ and verify that it transforms just like χ does.

3. A gauge field A_μ is called flat if $F_{\mu\nu} = 0$. Show that if a gauge field A is gauge transformation of the zero gauge field, i.e. if it has the form

$$A_\mu(x) = iU(x)\partial_\mu U^{-1}(x)$$

for some gauge transformation U, then it is flat. The converse is actually also true (on $\mathbb{R}^4$), but it is harder to prove. Deduce that a flat gauge field solves the Yang-Mills equations of motion $D_\mu F^{\mu\nu} = 0$. (Such solutions of course are rather trivial, in the sense that they are gauge-equivalent to the zero solution).