Week 2 (due April 16)

1. In this exercise you will apply the Faddeev-Popov procedure to simplify some finite-dimensional integrals over spaces of matrices. Such integrals are known as matrix integrals and have many applications in physics and mathematics.

(a) Consider an integral of the form

$$\int dM f(M),$$

where M is a Hermitian $N \times N$ matrix and $f(M)$ is a function invariant under arbitrary unitary transformations:

$$f(UMU^\dagger) = f(M).$$

Show that the usual measure

$$dM = \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}\, M_{ij} d\mathrm{Im}\, M_{ij}$$

is invariant under these transformations.

(b) We would like to reduce the matrix integral to the integral over the space of orbits of $U(N)$ action. Since every Hermitian matrix can be diagonalized by a unitary transformation, these orbits are labeled by the (unordered) eigenvalues of M, i.e. by N unordered real numbers. Compute the measure on this space using the Faddeev-Popov procedure. That is, consider the gauge-fixing conditions

$$M_{ij} = 0, \quad \forall i < j.$$

Also, consider the function $\Delta(M)$ defined by

$$\Delta(M) \int dU \prod_{i<j} \delta^2((UMU^\dagger)_{ij}) = 1.$$

The usual Faddeev-Popov manipulations show that the matrix integral $\int dM f(M)$ is equal to

$$\frac{1}{N!} \mathrm{vol}(U(N)) \int \prod_{i=1}^N dm_i \; \Delta(\mathrm{diag}(m_1, \ldots, m_N)) f(\mathrm{diag}(m_1, \ldots, m_N)).$$

The factor $1/N!$ arises from the fact that sets of eigenvalues related by a permutation label the same orbit. To get the measure on the eigenvalues $m_1, \ldots, m_N$, evaluate $\Delta(M)$ for a diagonal M.

(c) Repeat (b) for the case when the Hermitian matrix M is replaced by a real symmetric matrix, and the unitary group $U(N)$ is replaced by the orthogonal group $O(N)$. Orbits of $O(N)$ action on the space of real symmetric matrices are again labeled by N unordered real numbers, but the measure on this space is different.

(d) Repeat (b) for the integral

$$\int dV f(V),$$

where V is a unitary matrix, and $f(V)$ is invariant under conjugation. Note that $U(N)$ acts by conjugation on itself, and every unitary matrix V can be diagonalized by a $U(N)$ transformation. Thus we can label orbits of $U(N)$ action by the unordered eigenvalues of V, i.e. by N unordered complex numbers with absolute value 1. We can parameterize them as $\exp(i\alpha_j), j = 1, \ldots, N$, where α_j runs from 0 to 2π. Your task is to determine the measure on the space of α_j variables.

2. Compute the BRST current (i.e. the Noether current corresponding to the BRST symmetry) in the Yang-Mills theory without matter fields. Use the Lorenz gauge.