Week 2 (due April 16)

1. In this exercise you will apply the Faddeev-Popov procedure to simplify some finite-dimensional integrals over spaces of matrices. Such integrals are known as matrix integrals and have many applications in physics and mathematics.

(a) Consider an integral of the form
\[ \int dM f(M), \]
where \( M \) is a Hermitian \( N \times N \) matrix and \( f(M) \) is a function invariant under arbitrary unitary transformations:
\[ f(UMU^\dagger) = f(M). \]
Show that the usual measure
\[ dM = \prod_i dM_{ii} \prod_{i<j} d\text{Re} M_{ij} d\text{Im} M_{ij} \]
is invariant under these transformations.

(b) We would like to reduce the matrix integral to the integral over the space of orbits of \( U(N) \) action. Since every Hermitian matrix can be diagonalized by a unitary transformation, these orbits are labeled by the (unordered) eigenvalues of \( M \), i.e. by \( N \) unordered real numbers. Compute the measure on this space using the Faddeev-Popov procedure. That is, consider the gauge-fixing conditions
\[ M_{ij} = 0, \quad \forall i < j. \]
Also, consider the function \( \Delta(M) \) defined by
\[ \Delta(M) \int dU \prod_{i<j} \delta^2((UMU^\dagger)_{ij}) = 1. \]
The usual Faddeev-Popov manipulations show that the matrix integral \( \int dM f(M) \) is equal to
\[ \frac{1}{N! \text{vol}(U(N))} \int \prod_{i=1}^N dm_i \Delta(\text{diag}(m_1, \ldots, m_N)) f(\text{diag}(m_1, \ldots, m_N)). \]
The factor $1/N!$ arises from the fact that sets of eigenvalues related by a permutation label the same orbit. To get the measure on the eigenvalues $m_1, \ldots, m_N$, evaluate $\Delta(M)$ for a diagonal $M$.

(c) Repeat (b) for the case when the Hermitian matrix $M$ is replaced by a real symmetric matrix, and the unitary group $U(N)$ is replaced by the orthogonal group $O(N)$. Orbits of $O(N)$ action on the space of real symmetric matrices are again labeled by $N$ unordered real numbers, but the measure on this space is different.

(d) Repeat (b) for the integral

$$\int dV f(V),$$

where $V$ is a unitary matrix, and $f(V)$ is invariant under conjugation. Note that $U(N)$ acts by conjugation on itself, and every unitary matrix $V$ can be diagonalized by a $U(N)$ transformation. Thus we can label orbits of $U(N)$ action by the unordered eigenvalues of $V$, i.e. by $N$ unordered complex numbers with absolute value 1. We can parameterize them as $\exp(i\alpha_j), j = 1, \ldots, N$, where $\alpha_j$ runs from 0 to $2\pi$. Your task is to determine the measure on the space of $\alpha_j$ variables.

2. Compute the BRST current (i.e. the Noether current corresponding to the BRST symmetry) in the Yang-Mills theory without matter fields. Use the Lorenz gauge.