Week 5 (due May 7)

1. Consider the chiral Lagrangian describing Goldstone bosons in a theory with $SU(N)_L \times SU(N)_R$ global symmetry spontaneously broken down to the diagonal $SU(N)$:

$$L = -F^2 \mathrm{Tr}(\partial_\mu U \partial^\mu U^{-1}),$$

where F is a constant and the field $U(x) = \exp(iT^a \phi^a)$ takes values in $SU(N)$. Under the global $SU(N)_L \times SU(N)_R$ symmetry it transforms as

$$U(x) \mapsto V_L U(x) V_R^{-1}.$$

This model is also known as the principal chiral model.

(a) Write down the equations of motion for this theory. Hint: U is constrained to have a unit determinant, so one needs to add a Lagrange multiplier field λ to enforce the constraint $\det U = 1$. The field λ can later be eliminated using the she constraint to get a closed-form equation of motion for U only. The constraint $UU^\dagger = 1$ can either be imposed later, or from the start using another Lagrange multiplier field.

(b) Compute the conserved currents J_L^μ and J_R^μ associated with the symmetries $SU(N)_L$ and $SU(N)_R$. Check that they are indeed conserved on-shell. Show that the currents transform in the adjoint representations of the respective symmetries.

(c) Parity symmetry $\mathbf{x} \to -\mathbf{x}$ acts on the symmetry group by swapping $SU(N)_L \times SU(N)_R$ and on $U(x)$ by

$$U(x^0, \mathbf{x}) \to U(x^0, -\mathbf{x})^{-1}.$$

The above Lagrangian is obviously invariant under this symmetry. However, it is also invariant under the naive parity

$$P_{naive} : U(x^0, \mathbf{x}) \to U(x^0, -\mathbf{x}),$$

as well as an internal symmetry

$$G : U(x) \mapsto U(x)^{-1}.$$

If we parameterize $U(x)$ as $\exp(iT^a \phi^a(x))$, where T^a are the generators of $SU(N)$ and $\phi^a(x)$ are real scalars, then the symmetry G acts simply by $\phi^a \mapsto -\phi^a$. This means that all Green's functions with an odd number of

Goldstone bosons will vanish identically. This is worrisome, since (1) neither naive parity nor G-parity are present in QCD; (2) even worse, processes involving an odd number of Goldstone bosons are observed, and their rate is not suppressed relative to the rate of other nuclear processes. So it looks like we are missing some important term in the action. This term is known as the Wess-Zumino-Witten term and is rather complicated. Instead of looking for such a term, can you find a modification of the equations of motion which breaks G and the naive parity? (Hint: since the extra term should break the naive parity, it should involve the antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. It should also transform under $SU(N)_L \times SU(N)_R$ symmetry in the same way as the other terms in the equations of motion.)

(d) Now specialize to the case $N = 2$. Then T^a, $a = 1, 2, 3$, are the usual Pauli matrices divided by 2. Use the known properties of the Pauli matrices to express the Lagrangian in terms of the three scalar fields ϕ^a, $a = 1, 2, 3$ which describe pions.

(e) The $SU(2)$ group manifold is the same as S^3. On the other hand, one can use the stereographic projection to identify S^3 minus a single point with $\mathbb{R}^3$. The usual round metric on S^3 then looks like

$$\frac{\sum_{a=1}^{3} dx^a dx^a}{(1 + |x|^2)^2},$$

where $|x|^2 = \sum_a x^a x^a$. Hence we expect that the theory of a scalar field taking values in $SU(2)$ can be written in terms of three scalars $\psi^a(x)$, so that the Lagrangian is proportional to

$$\frac{\partial_\mu \psi^a \partial^\mu \psi^a}{(1 + \psi^b \psi^b)^2}.$$

Relate $\psi^a(x)$ with the fields $\phi^a(x)$ used in part (d). (The physics, of course, does not change after a field redefinition.)