Week 7 (due Feb. 26)

1. Using the Poisson brackets for electric and magnetic fields in Maxwell theory (derived in class) and the Hamiltonian

$$H = \int d^3x \frac{1}{2}(E^2 + B^2),$$

derive the Hamiltonian equations of motion and show that they are equivalent to six out of eight Maxwell equations. (The missing two equations are the Gauss law $\nabla \cdot E = 0$ and its magnetic analogue $\nabla \cdot B = 0$. They should be imposed separately because they are not dynamic equations but constraints on the initial data).

2. Consider the theory of a gauge field A_μ in three-dimensional space-time with a Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}k\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The second term in the Lagrangian is called the Chern-Simons term, and the theory is called Chern-Simons-Maxwell theory.

(a) Show that the action $S = \int L d^3x$ is gauge-invariant. Derive the equations of motion for the field A. Show that they imply that each component of the tensor $F_{\mu\nu}$ satisfies the massive Klein-Gordon equation for some mass m. Express m in terms of k. How many polarization states to these "massive waves" have?

(bc) In three dimensions the analog of the electric field $E_i = -\partial_0 A_i + \partial_i A_0$ is a vector, but the analog of the magnetic field $B = \epsilon_{ij}\partial_i A_j$ is a scalar. As in four dimensions, A_0 should be thought of as a Lagrange multiplier whose equation of motion gives a constraint (the analog of the Gauss law) and does not have a conjugate momentum, while the fields A_i, $i = 1, 2$, do. What is the analog of the Gauss law constraint $\partial_i E_i = 0$ in this theory ? Find the momenta conjugate to A_i. Write the Hamiltonian in terms of B and the momenta conjugate to A_i. Write the action in Hamiltonian form (i.e. in a form where the momenta and A_i are regarded as independent and the action is linear in time derivatives).

(d) Compute the Poisson brackets of E_i and B.

(e) Compute the propagator for A in the Lorenz gauge $\partial^\mu A_\mu = 0$.