1 Free real scalar field

The Hamiltonian is

$$H = \int d^3x \mathcal{H} = \frac{1}{2}\int d^3x \left(p(x)^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

Let us expand both ϕ and p in Fourier series:

$$\phi(t,\mathbf{x}) = \int \frac{d^3p}{2\omega(\mathbf{p})}\tilde{\phi}(t,\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}, \quad p(t,\mathbf{x}) = \int \frac{d^3p}{2\omega(p)}\tilde{p}(t,\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}.$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$. Then:

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3(2\omega(\mathbf{p}))^2}\left(|\tilde{p}(\mathbf{p})|^2 + |\tilde{\phi}(\mathbf{p})|^2\omega(\mathbf{p})^2\right).$$

This is a Hamiltonian for an infinite collection of harmonic oscillators labeled by $\mathbf{p} \in \mathbb{R}^3$ and energy $\omega(p)$. Introduce creation-annihilation operators:

$$a(\mathbf{p}) = \frac{p(\mathbf{p}) - i\phi(\mathbf{p})}{\omega(\mathbf{p})\sqrt{2}}, \quad a^\dagger(\mathbf{p}) = \frac{p^\dagger(\mathbf{p}) + i\phi^\dagger(\mathbf{p})}{\omega(\mathbf{p})\sqrt{2}}.$$

Then:

$$H = \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})}\omega(\mathbf{p})\left(a^\dagger(\mathbf{p})a(\mathbf{p}) + \frac{1}{2}\right)$$

The last term in parentheses can be dropped (divergent vacuum energy). The operators $a, a^\dagger$ satisfy:

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^3 2\omega(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}'), \quad [a(\mathbf{p}), a(\mathbf{p}')] = 0.$$

The expression $2\omega(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')$ is Lorenz-invariant, so this is a natural normalization of creation-annihilation operators in a relativistic theory.

So, as expected, the free scalar field describes noninteracting spinless bosonic particles with a relativistic energy-momentum relation $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$.

2 Free complex scalar field

Commutation relations:

$$
\begin{aligned}
[\phi(t,\mathbf{x}), p(t,\mathbf{y})] &= i\delta^3(\mathbf{x}-\mathbf{y}),\\
[\phi(t,\mathbf{x})^\dagger, p(t,\mathbf{y})^\dagger] &= i\delta^3(\mathbf{x}-\mathbf{y}),\\
[\phi(t,\mathbf{x})^\dagger, p(t,\mathbf{y})] &= 0,\\
[\phi(t,\mathbf{x}), p(t,\mathbf{y})^\dagger] &= 0,\\
[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] &= 0,\\
[\phi(t,\mathbf{x})^\dagger, \phi(t,\mathbf{y})^\dagger] &= 0,\\
[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})^\dagger] &= 0,\\
[p(t,\mathbf{x}), p(t,\mathbf{y})] &= 0,\\
[p(t,\mathbf{x})^\dagger, p(t,\mathbf{y})^\dagger] &= 0,\\
[p(t,\mathbf{x}), p(t,\mathbf{y})^\dagger] &= 0.
\end{aligned}
$$

Here $p = \dot{\phi}^\dagger, p^\dagger = \dot{\phi}$.
Hamiltonian:

$$H = \int d^3x \left(pp^\dagger + \partial_i\phi^\dagger\partial_i\phi + m^2\phi^\dagger\phi \right).$$

Let us show that these equations describe the bosonic Fock space for relativistic particles (with $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$). Let us Fourier transform the scalar field ϕ:

$$\phi(t,\mathbf{x}) = \int \frac{d^3p}{2E_\mathbf{p}(2\pi)^3}\tilde{\phi}(t,\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}.$$

The Klein-Gordon equation

$$(\partial_0^2 - \nabla^2 + m^2)\phi = 0$$

gives an ordinary differential equation for $\tilde{\phi}(t,\mathbf{p})$:

$$\frac{\partial^2\tilde{\phi}}{\partial t^2} = -(\mathbf{p}^2 + m^2)\tilde{\phi}.$$

The general solution is

$$\tilde{\phi}(t,\mathbf{p}) = e^{-iE_pt}a(\mathbf{p}) + e^{iE_pt}c(\mathbf{p}).$$

It will be convenient to rename $c(\mathbf{p}) = b(-\mathbf{p})^\dagger$. Then

$$\phi(t,\mathbf{x}) = \int \frac{d^3p}{2E_\mathbf{p}(2\pi)^3}\left(a_\mathbf{p} e^{ip\cdot x} + b(\mathbf{p})^\dagger e^{-ip\cdot x}\right).$$

Similarly

$$\phi(t,\mathbf{x})^\dagger = \int \frac{d^3p}{2E_\mathbf{p}(2\pi)^3}\left(b_\mathbf{p} e^{ip\cdot x} + a(\mathbf{p})^\dagger e^{-ip\cdot x}\right).$$

We can invert these formulas and express $a, b, a^\dagger, b^\dagger$ in terms of $\phi, \dot\phi$ and $\phi^\dagger, \dot\phi^\dagger$. (This is an exercise). Then the commutation relations of $a, a^\dagger, b, b^\dagger$ turn out

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3 2E_\mathbf{p}\delta^3(\mathbf{p}-\mathbf{q}), \tag{1}$$

$$[b(\mathbf{p}), b^\dagger(\mathbf{q})] = (2\pi)^3 2E_\mathbf{p}\delta^3(\mathbf{p}-\mathbf{q}), \tag{2}$$

with all other commutators vanishing. Thus it is natural to postulate the existence of the vacuum state $|0\rangle$, annihilated by all $a(\mathbf{p})$ and $b(\mathbf{p})$. Then the Hilbert space is the bosonic Fock space built on the sum of two copies of $L^2(\mathbb{R}^3)$. Why two copies? We expected only one! Resolution: we have an additional quantum number which distinguishes b-particles from a-particles. The b-particles are actually anti-particles of a-particles! (see below).

Hamiltonian becomes

$$H = \frac{1}{2}\int \frac{d^3p}{2E_\mathbf{p}(2\pi)^3} E_\mathbf{p}\left(a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p}) + b(\mathbf{p})b^\dagger(\mathbf{p})\right).$$

Let us normal-order it:

$$H = V(2\pi)^{-3}\int d^3p E_\mathbf{p} + \ldots$$

Thus the vacuum energy density is divergent. If we cut off the integral at $|\mathbf{p}| = \Lambda$, we find

$$\mathcal{E}_0 = \frac{\Lambda^4}{8\pi^2}$$

This is the simplest example of an *ultraviolet divergence.*

3 Noether's theorem

(Reading: section 22, pp. 132-135).

Noether's theorem says that for every continuous symmetry of the action there is a current j_μ (vector-valued function made of fields and their derivatives) which satisfies

$$\partial_\mu j^\mu = 0.$$

This implies that

$$Q = \int d^3x j^0(t, \mathbf{x})$$

is time-independent. I.e. it is a conserved charge. In the Hamiltonian formalism this is expressed as $Q, H = 0$, which upon quantization becomes $[Q, H] = 0$.

Let us derive the Noether theorem for a theory of scalar fields with a Lagrangian $\mathcal{L}(\phi^a)$. Suppose the infinitesimal symmetry transformation is given by

$$\delta\phi^a = \epsilon \cdot v^a(\phi).$$

Consider now the same transformation, but with ϵ a function of x. Since the action is of first order in derivatives of ϕ, the variation of the action must be of the form

$$\delta S = \int d^4x j^\mu \partial_\mu \epsilon,$$

for some j^μ independent of ϵ. But on equations of motion this must vanish, for arbitrary ϵ. Therefore $\partial_\mu j^\mu = 0$.

Let us apply this procedure to the complex scalar field ϕ and the transformation

$$\delta\phi = i\epsilon\phi, \quad \delta\phi^\dagger = -i\epsilon\phi^\dagger.$$

The variation of the action is

$$\delta S = i \int d^4x \partial_\mu \epsilon \left(-\phi^\dagger \partial^\mu \phi + \partial^\mu \phi^\dagger \phi \right).$$

Hence the current is

$$j_\mu = -i \left(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi \right).$$

What is the meaning of the corresponding charge, in terms of particles?

$$Q = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) - b^\dagger(\mathbf{k}) b(\mathbf{k}) \right).$$

I.e. it is the number of particles minus the number of anti-particles.

Let me consider another example: translational symmetry. Here

$$\delta\phi = \epsilon^\mu \partial_\mu \phi.$$

Note that here ϵ has a vector index. Thus we expect

$$\delta S = \int d^4x \partial_\nu \epsilon^\mu T^\nu_\mu$$

for some tensor T. (It is called the stress-energy tensor). Let us determine T. For constant ϵ we have

$$\delta S = \int d^4x \epsilon^\mu \partial_\mu \mathcal{L}.$$

This indeed vanishes for constant ϵ (by integration by parts), but does not vanish for nonconstant ϵ. But for non-constant ϵ we also get other terms in the variation:

$$\delta S = \int d^4x \left(-\partial_\mu \epsilon^\mu \mathcal{L} + \partial_\mu \epsilon^\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi \right).$$

Hence

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}.$$

For the free scalar field, we get

$$T^\mu_\nu = -\partial^\mu \phi^\dagger \partial_\nu \phi + \partial^\mu \phi \partial_\nu \phi^\dagger - \delta^\mu_\nu \mathcal{L}.$$

For example:

$$T^0_0 = \partial_0 \phi^\dagger \partial_0 \phi + \nabla \phi^\dagger \nabla \phi + m^2 \phi^\dagger \phi.$$

The corresponding "charge" is the energy (i.e. the Hamiltonian). Similarly,

$$T^0_i = \partial_0 \phi^\dagger \partial_i \phi + \partial_i \phi^\dagger \partial_0 \phi.$$

The corresponding charge is minus the momentum. Indeed, after expressing in terms of a and b get

$$\int d^3x T^0_i = -\int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} k_i \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right).$$

Starting from a symmetry, one can get a conserved charge. Conversely, starting from a conserved charge Q, one can try to get a symmetry transformation, by letting

$$\delta F = \{Q, F\}.$$

Then $\delta H = 0$, and δ commutes with time translations.

One can show directly that Q is the generator of symmetry transformations:

$$Q = -\int d^3x p_i \delta\phi^i, \quad \{Q, \phi^j\} = \delta\phi^j.$$

In quantum theory:

$$[Q, \phi^j] = -i\delta\phi^j.$$

A finite transformation is

$$\phi \to U^{-1}\phi U, \quad U = \exp(-itQ).$$

In relativistic field theory, we are interested in translations and Lorenz transformations. Together they form Poincare group:

$$x \to \Lambda x + a.$$

Generators of translations are momenta $P_\mu = \int d^3x T^0_\mu$. Lorenz transformations act by

$$\phi'(x) = \phi(\Lambda^{-1}x).$$

Infinitesimal transformation $\Lambda = 1 + \omega$ gives

$$\delta\phi = \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi.$$

We can achieve this by letting

$$M^{\mu\nu} = \int d^3x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right).$$

This suggests that the conserved current for the Lorenz transformations is

$$L^{\rho\mu\nu} = x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu}.$$

It is conserved because $T^{\mu\nu} = T^{\nu\mu}$.

It is interesting to compute Poisson brackets or commutator of all these generators. For example:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma).$$

This algebra characterizes infinitesimal Lorenz transformations. Infinitesimal rotations are

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk},$$

infinitesimal boosts are $K_i = M^{i0}$. In terms of J and K we have

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_j, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

The other commutators are

$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\sigma}P^\rho - (\rho \leftrightarrow \sigma)).$$

4 Wick theorem

Observables are polynomial functions of ϕ and its derivatives. Need an efficient way to evaluate vacuum expectation values of such observables.

Wick theorem helps us do this. Let f_α, $\alpha = 1, \ldots, N$ be a linear function of $a_i, a_i^\dagger$, where i is either a discrete or continuous index. Then

$$\langle 0|f_1 \ldots f_N|0\rangle = \frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_N}|_{\lambda_\alpha=0} \exp\left(\frac{1}{2}\sum_{\alpha\beta}\lambda_\alpha\lambda_\beta\Delta_{\alpha\beta}\right),$$

where

$$\Delta_{\alpha\beta} = \langle 0|f_\alpha f_\beta|0\rangle.$$

This can be written in terms of "pairings" of $f_1, \ldots, f_N$. Note that the vacuum expectation value vanishes if N is odd.

Proof proceeds as follows. First evaluate

$$\langle 0|\exp(\sum_i(\lambda_i a_i + \bar{\lambda}a_i^\dagger))|0\rangle = \exp(\frac{1}{2}\sum_i \bar{\lambda}_i\lambda_i).$$

This follows from the Baker-Campbell-Hausdorff formula. Then re-express the r.h.s. in terms of expectation values.

5 The spin-statistics relation

Let us compute the commutator of $\phi(x)$ and $\phi(y)$ (in the real case). It vanishes outside the light-cone.

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left(e^{ik(x-y)} - e^{-ik(x-y)}\right).$$

Clearly Lorenz-invariant, so is a function only of $(x-y)^2$ and maybe $x^0 - y^0$ (if $(x-y)^2 < 0$).

Now suppose the separation is space-like. Can make $(x-y)$ purely spatial. Then the integrand is odd, and the commutator vanishes. For a time-like separation one gets something nonzero.

What if we tried to declare a and $a^\dagger$ fermionic oscillators instead? Would get cos instead of sin, so the anti-commutator would be non-vanishing. (The commutator would be even worse).

6 Scattering theory

First:

$$\int d^3x e^{-ikx}\phi(x) = \frac{1}{2\omega}a(\mathbf{k}) + \frac{1}{2\omega}e^{2i\omega t}a^\dagger(-\mathbf{k}),$$

$$\int d^3x e^{-ikx}\partial_0\phi = -\frac{i}{2}a(\mathbf{k}) + \frac{i}{2}e^{2i\omega t}a^\dagger(-\mathbf{k}).$$

Hence

$$a(\mathbf{k}) = \int d^3x e^{-ikx}\left(i\partial_0\phi + \omega\phi\right) = i\int d^3x e^{-ikx}\overleftrightarrow{\partial_0}\phi.$$