1. (a)

It is simplest to first evaluate $[\hat{N}, \Psi^\dagger(y)]$:

$$[\hat{N}, \Psi^\dagger(y)] = \int \mathrm{d}^3x [\Psi^\dagger(x)\Psi(x), \Psi^\dagger(y)]. \tag{1}$$

If we remember the relations

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}, \tag{2}$$
$$[\Psi^\dagger(x), \Psi^\dagger(y)] = [\Psi(x), \Psi(y)] = 0, \tag{3}$$
$$[\Psi(x), \Psi^\dagger(y)] = \delta^3(x-y), \tag{4}$$

then we have from (1)

$$\begin{aligned}[\hat{N}, \Psi^\dagger(y)] &= \int \mathrm{d}^3x \{\Psi^\dagger(x)[\Psi(x), \Psi^\dagger(y)] + [\Psi^\dagger(x), \Psi^\dagger(y)]\Psi(x)\} \\ &= \int \mathrm{d}^3x \Psi^\dagger(x)\delta^3(x-y) = \boxed{\Psi^\dagger(y).}\end{aligned} \tag{5}$$

This also tells us

$$[\hat{N}, \Psi(y)] = [\Psi^\dagger(y), \hat{N}^\dagger]^\dagger = -[\hat{N}, \Psi^\dagger(y)]^\dagger = -\Psi(y). \tag{6}$$

The physical meaning of (5) can be understood by applying it to a state with n particles:

$$\begin{aligned}[\hat{N}, \Psi^\dagger(x)] |n\rangle &= \Psi^\dagger(x) |n\rangle \\ [\hat{N}\Psi^\dagger(x) - \Psi^\dagger(x)\hat{N}] |n\rangle &= \\ [\hat{N}\Psi^\dagger(x) - n\Psi^\dagger(x)] |n\rangle &= \\ \hat{N}\Psi^\dagger(x) |n\rangle &= (n+1)\Psi^\dagger(x) |n\rangle .\end{aligned} \tag{7}$$

This demonstrates that $\Psi^\dagger(x) |n\rangle$ is a state with $n+1$ particles, so $\Psi^\dagger(x)$ increases the number of particles in a state by one (it creates a particle).

With (5) and (6), we can evaluate $[\hat{H}_0, \hat{N}]$:

$$\begin{aligned}[\hat{H}_0, \hat{N}] &= \int \mathrm{d}^3x [\Psi^\dagger(x)\nabla^2\Psi(x), \hat{N}] \\ &= \int \mathrm{d}^3x \{\Psi^\dagger(x)[\nabla^2\Psi(x), \hat{N}] + [\Psi^\dagger(x), \hat{N}]\nabla^2\Psi(x)\} \\ &= \int \mathrm{d}^3x \{\Psi^\dagger(x)\nabla^2\Psi(x) - \Psi^\dagger(x)\nabla^2\Psi(x)\} = \boxed{0.}\end{aligned} \tag{8}$$

The commutator of an operator $\hat{Q}$ with the Hamiltonian dictates how $\hat{Q}$ evolves in time by $\frac{d\hat{Q}}{dt} = i[\hat{H}_0, \hat{Q}]$. Thus, (8) implies that the number of particles is conserved, i.e. the number of particles does not change with time.

1. (b)

It is important to note that position is no longer treated as an operator in the formalism we are using. Instead, it is merely a label. Anyway, if we remember that $\int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{i(q-p)x} = \delta^3(p-q)$, we can rewrite $\hat{N}$ in terms of $b^\dagger(p)$ and $b(p)$.

$$\begin{aligned}
\hat{N} &= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} b^\dagger(p) e^{-ipx} \int \frac{\mathrm{d}^3q}{(2\pi)^3} b(q) e^{iqx} \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} b^\dagger(p) b(q) \int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{i(q-p)x} \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} b^\dagger(p) b(q) \delta^3(p-q) \\
&= \boxed{\int \frac{\mathrm{d}^3p}{(2\pi)^3} b^\dagger(p) b(p).}
\end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned}
\hat{H}_0 &= \frac{-1}{2m} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} b^\dagger(p) e^{-ipx} \nabla^2 \int \frac{\mathrm{d}^3q}{(2\pi)^3} b(q) e^{iqx} \\
&= \frac{-1}{2m} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} b^\dagger(p) e^{-ipx} \int \frac{\mathrm{d}^3q}{(2\pi)^3} b(q) \nabla^2 e^{iqx} \\
&= \frac{-1}{2m} \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} b^\dagger(p)(-q^2) b(q) \int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{i(q-p)x} \\
&= \boxed{\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{p^2}{2m} b^\dagger(p) b(p).}
\end{aligned} \tag{10}$$

Notice that on the second line of (10), ∇^2 only acts on e^{iqx} because the other terms do not depend on the label x.

1. (c)

There are several ways to do this. I'll proceed by noting that $b(p)$ must be the inverse Fourier transform of $\Psi(x)$:

$$b(p) = \int \mathrm{d}^3 x \Psi(x) e^{-ipx}, \tag{11}$$

$$b^\dagger(p) = \int \mathrm{d}^3 x \Psi^\dagger(x) e^{ipx}. \tag{12}$$

Then

$$[\hat{N}, b^\dagger(p)] = \int \mathrm{d}^3 x [\hat{N}, \Psi^\dagger(x)] e^{ipx} = \int \mathrm{d}^3 x \Psi^\dagger(x) e^{ipx} = \boxed{b^\dagger(p).} \tag{13}$$

We can also compute the commutators of $b(p)$ and $b^\dagger(p)$:

$$[b(p), b(q)] = \int \mathrm{d}^3 x \, \mathrm{d}^3 y [\Psi(x), \Psi(y)] e^{-ipx-iqy} = 0, \tag{14}$$

$$[b^\dagger(p), b^\dagger(q)] = \int \mathrm{d}^3 x \, \mathrm{d}^3 y [\Psi^\dagger(x), \Psi^\dagger(y)] e^{ipx+iqy} = 0, \tag{15}$$

$$\begin{aligned} [b(p), b^\dagger(q)] &= \int \mathrm{d}^3 x \, \mathrm{d}^3 y [\Psi(x), \Psi^\dagger(y)] e^{ipx-iqy} \\ &= \int \mathrm{d}^3 x \, \mathrm{d}^3 y \delta^3(x-y) e^{ipx-iqy} \\ &= \int \mathrm{d}^3 x e^{i(p-q)x} = (2\pi)^3 \delta^3(p-q). \end{aligned} \tag{16}$$

With the above results, we can find our answer directly:

$$\begin{aligned} [\hat{H}_0, b^\dagger(p)] &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{q^2}{2m} [b^\dagger(q) b(q), b^\dagger(p)] \\ &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{q^2}{2m} b^\dagger(q) [b(q), b^\dagger(p)] \\ &= \int \mathrm{d}^3 q \frac{q^2}{2m} b^\dagger(q) \delta^3(p-q) = \boxed{\frac{p^2}{2m} b^\dagger(p).} \end{aligned} \tag{17}$$

(13) and (17) can be understood in the same way that we understood (5). To be precise, if we apply $[\hat{N}, b^\dagger(p)]$ to a state with definite particle number,

we see that $b^\dagger(p)$ increases the particle number by 1. Similarly, if we apply $[\hat{H}_0, b^\dagger(p)]$ to a state with definite energy, we see that $b^\dagger(p)$ increases the energy by $\frac{p^2}{2m}$:

$$\begin{aligned}
[\hat{H}_0, b^\dagger(p)] \left|E\right\rangle &= \frac{p^2}{2m} b^\dagger(p) \left|E\right\rangle \\
[\hat{H}_0 b^\dagger(p) - b^\dagger(p)\hat{H}_0] \left|E\right\rangle &= \\
[\hat{H}_0 b^\dagger(p) - E b^\dagger(p)] \left|E\right\rangle &= \\
\hat{H}_0 b^\dagger(p) \left|E\right\rangle &= \left(E + \frac{p^2}{2m}\right) b^\dagger(p) \left|E\right\rangle .
\end{aligned} \tag{18}$$

This implies that $b^\dagger(p)$ creates a particle with energy $\frac{p^2}{2m}$.

2. (a)

The Baker-Campbell-Housedorf formula will be helpful for computing the norm:

$$\exp(A)\exp(B) = \exp(A + B + \frac{1}{2}[A, B] + \dots), \tag{19}$$

where the ellipsis indicates terms which involve commutators of commutators. Let us also define

$$\left|\Omega\right\rangle = \exp\left(\sum_i a_i^\dagger \lambda_i\right) \left|0\right\rangle . \tag{20}$$

The requirement of finite norm implies

$$\begin{aligned}
\left\langle\Omega|\Omega\right\rangle &= \prod_{i,j} \left\langle 0\right| \exp\left(a_i \lambda_i^* + a_j^\dagger \lambda_j + \frac{1}{2}[a_i \lambda_i^*, a_j^\dagger \lambda_j]\right) \left|0\right\rangle \\
&= \prod_{i,j} \exp\left(\frac{1}{2}\delta_i^j \lambda_i^* \lambda_j\right) \left\langle 0\right| \exp\left(a_j^\dagger \lambda_j + a_i \lambda_i^*\right) \left|0\right\rangle \\
&= \prod_{i,j} \exp\left(\frac{1}{2}\delta_i^j \lambda_i^* \lambda_j - \frac{1}{2}[a_j^\dagger \lambda_j, a_i \lambda_i^*]\right) \left\langle 0\right| \exp(a_j^\dagger \lambda_j) \exp(a_i \lambda_i^*) \left|0\right\rangle \\
&= \prod_{i,j} \exp\left(\delta_i^j \lambda_i^* \lambda_j\right) \left\langle 0|0\right\rangle \\
&= \exp\left(\sum_i |\lambda_i|^2\right) < \infty,
\end{aligned} \tag{21}$$

which in turn implies

$$\boxed{\sum_i |\lambda_i|^2 < \infty.} \tag{22}$$

We used the fact that the commutator of a and $a^\dagger$ is just a number, so it commutes with everything and thus the Baker-Campbell-Hausedorf formula terminates. Assume that we have chosen to normalize the state as $\langle\Omega|\Omega\rangle = C$, where C is a constant. Then

$$\langle N\rangle = \frac{\langle\Omega|\sum_i a_i^\dagger a_i|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \frac{1}{C}\sum_i \langle\Omega|\left(a_i a_i^\dagger - 1\right)|\Omega\rangle\,. \tag{23}$$

Now, we can use a trick to simplify this calculation. Notice that

$$a_i^\dagger|\Omega\rangle = \frac{d}{d\lambda_i}\exp\left(\sum_i a_i^\dagger\lambda_i\right)|0\rangle = \frac{d}{d\lambda_i}|\Omega\rangle\,. \tag{24}$$

The validity of this formula can be checked by writing out the exponential as a power series. Given this, we have

$$\begin{aligned}
\langle N\rangle &= \frac{1}{C}\sum_i\left(\frac{d}{d\lambda_i^*}\frac{d}{d\lambda_i} - 1\right)\langle\Omega|\Omega\rangle \\
&= \frac{1}{C}\sum_i\left(\frac{d}{d\lambda_i^*}\frac{d}{d\lambda_i} - 1\right)\exp\left(\sum_j|\lambda_j|^2\right) \\
&= \frac{1}{C}\sum_i\left(\frac{d}{d\lambda_i^*}\lambda_i^* - 1\right)\exp\left(\sum_j|\lambda_j|^2\right) \\
&= \frac{1}{C}\sum_i\left(|\lambda_i|^2 + 1 - 1\right)\exp\left(\sum_j|\lambda_j|^2\right) \\
&= \sum_i|\lambda_i|^2 = \boxed{\ln\langle\Omega|\Omega\rangle.}
\end{aligned} \tag{25}$$

To compute the standard deviation in the particle number, we should use

$$\sigma^2 = \left\langle N^2\right\rangle - \langle N\rangle^2\,. \tag{26}$$

Then

$$\begin{aligned}
\left\langle N^2 \right\rangle &= \langle\Omega| \frac{1}{C} \sum_i a_i^\dagger a_i \sum_j a_j^\dagger a_j |\Omega\rangle \\
&= \frac{1}{C} \sum_i \sum_j \langle\Omega| (a_i a_i^\dagger - 1)(a_j a_j^\dagger - 1) |\Omega\rangle \\
&= \frac{1}{C} \sum_i \sum_j \langle\Omega| [a_i(a_j a_i^\dagger - \delta_i^j) a_j^\dagger - a_i a_i^\dagger - a_j a_j^\dagger + 1] |\Omega\rangle \\
&= \frac{1}{C} \sum_i \sum_j \langle\Omega| (a_i a_j a_i^\dagger a_j^\dagger - \delta_i^j a_i a_j^\dagger - a_i a_i^\dagger - a_j a_j^\dagger + 1) |\Omega\rangle
\end{aligned} \tag{27}$$

Let's take this one term at a time.

$$\begin{aligned}
\langle\Omega| a_i a_j a_i^\dagger a_j^\dagger |\Omega\rangle &= \frac{\partial}{\partial\lambda_i^*} \frac{\partial}{\partial\lambda_j^*} \frac{\partial}{\partial\lambda_i} \frac{\partial}{\partial\lambda_j} \langle\Omega|\Omega\rangle \\
&= \frac{\partial}{\partial\lambda_i^*} \frac{\partial}{\partial\lambda_j^*} \lambda_i^* \lambda_j^* \langle\Omega|\Omega\rangle \\
&= \frac{\partial}{\partial\lambda_i^*} (\delta_i^j \lambda_j^* + \lambda_i^* + \lambda_i^* |\lambda_j|^2) \langle\Omega|\Omega\rangle \\
&= (\delta_i^j \delta_i^j + \delta_i^j \lambda_j^* \lambda_i + 1 + |\lambda_i|^2 + |\lambda_j|^2 + \delta_i^j \lambda_i^* \lambda_j + |\lambda_i|^2 |\lambda_j|^2) \langle\Omega|\Omega\rangle \,,
\end{aligned} \tag{28}$$

$$\begin{aligned}
\langle\Omega| \delta_i^j a_i a_j^\dagger |\Omega\rangle &= \delta_i^j \frac{\partial}{\partial\lambda_i^*} \frac{\partial}{\partial\lambda_j} \langle\Omega|\Omega\rangle \\
&= \delta_i^j \frac{\partial}{\partial\lambda_i^*} \lambda_j^* \langle\Omega|\Omega\rangle \\
&= (\delta_i^j \delta_i^j + \delta_i^j \lambda_j^* \lambda_i) \langle\Omega|\Omega\rangle \,,
\end{aligned} \tag{29}$$

and from before

$$\langle\Omega| a_i a_i^\dagger |\Omega\rangle = (|\lambda_i|^2 + 1) \langle\Omega|\Omega\rangle \,. \tag{30}$$

Taking (28), (29), and (30) together, we get

$$\begin{aligned}
\left\langle N^2 \right\rangle &= \frac{1}{C} \sum_i \sum_j (\delta_i^j \lambda_i^* \lambda_j + |\lambda_i|^2 |\lambda_j|^2) \langle\Omega|\Omega\rangle \\
&= \ln C + (\ln C)^2.
\end{aligned} \tag{31}$$

Finally, we have

$$\sigma^2 = \ln C + (\ln C)^2 - (\ln C)^2 = \ln C, \tag{32}$$

which implies that the standard deviation is given by

$$\boxed{\sigma = \sqrt{\ln \langle \Omega | \Omega \rangle}.} \tag{33}$$

Notice that (25) and (33) provide evidence that the distribution function for the particle number is a Poisson distribution.

2. (b)

Since ordinary numbers are guaranteed to commute with operators, the fact that b_i and $b_i^\dagger$ have the same commutation relations as a_i and $a_i^\dagger$ follows immediately. Before we prove that $|\Omega\rangle$ is the vacuum for these new operators, it will be convenient to prove a couple of lemmas. First:

$$[a_i, (a_j^\dagger)^n] = n(a_j^\dagger)^{n-1}\delta_i^j. \tag{34}$$

We can prove this through induction. It clearly holds for the $n = 1$ case. We should then prove that it works for the $n = k + 1$ case under the assumption that it holds for the $n = k$ case:

$$\begin{aligned} [a_i, (a_j^\dagger)^{k+1}] &= [a_i, a_j^\dagger (a_j^\dagger)^k] \\ &= [a_i, a_j^\dagger](a_j^\dagger)^k + a_j^\dagger [a_i, (a_j^\dagger)^k] \\ &= \delta_i^j (a_j^\dagger)^k + a_j^\dagger k (a_j^\dagger)^{k-1} \delta_i^j \\ &= (k+1)(a_j^\dagger)^k \delta_i^j. \end{aligned} \tag{35}$$

(34) then implies the following general result:

$$[a_i, f(a_j^\dagger)] = \frac{\partial f}{\partial a_j^\dagger}\delta_i^j \tag{36}$$

whenever f can be expanded as a power series in $a_j^\dagger$ (since we can just differentiate term by term). Note that we could have used this trick as an alternative route for our calculations in (a).

Using (36), let's show that $|\Omega\rangle$ is the vacuum state for these new raising and lowering operators.

$$b_i |\Omega\rangle = (a_i - \lambda_i) \exp\left(\sum_j a_j^\dagger \lambda_j\right) |0\rangle . \tag{37}$$

Now, note that

$$\begin{aligned} a_i \exp\left(\sum_j a_j^\dagger \lambda_j\right) |0\rangle &= \left[[a_i, \exp\left(\sum_j a_j^\dagger \lambda_j\right)] + \exp\left(\sum_j a_j^\dagger \lambda_j\right) a_i \right] |0\rangle \\ &= \frac{\partial \exp\left(\sum_j a_j^\dagger \lambda_j\right)}{\partial a_j^\dagger} \delta_i^j |0\rangle \\ &= \lambda_i \exp\left(\sum_j a_j^\dagger \lambda_j\right) |0\rangle . \end{aligned} \tag{38}$$

Therefore

$$b_i |\Omega\rangle = (\lambda_i - \lambda_i) \exp\left(\sum_j a_j^\dagger \lambda_j\right) |0\rangle = \boxed{0,} \tag{39}$$

which means that $|\Omega\rangle$ is the new vacuum for these operators.

2. (c)

It's not too hard to guess the correct answer by looking at the original expression for the Hamiltonian:

$$\boxed{b_i = a_i + \frac{\beta_i^*}{\omega_i},} \tag{40}$$

$$\boxed{b_i^\dagger = a_i^\dagger + \frac{\beta_i}{\omega_i}.} \tag{41}$$

Then

$$\sum_i \omega_i b_i^\dagger b_i = \sum_i \left(\omega_i a_i^\dagger a_i + \beta_i a_i + \beta_i^* a_i^\dagger + \frac{|\beta_i|^2}{\omega_i} \right) , \tag{42}$$

which indicates that $\boxed{E_0 = -\sum_i \frac{|\beta_i|^2}{\omega_i}.}$

2. (d)

Let's just do the computation:

$$\begin{aligned}
[b, b^\dagger] &= [a\cosh t + a^\dagger \sinh t, a^\dagger \cosh t + a \sinh t] \\
&= [a\cosh t, a^\dagger \cosh t] + [a^\dagger \sinh t, a \sinh t] \\
&= \cosh^2 t - \sinh^2 t = 1,
\end{aligned} \tag{43}$$

$$\begin{aligned}
[b, b] &= [a\cosh t + a^\dagger \sinh t, a \cosh t + a^\dagger \sinh t] \\
&= [a\cosh t, a^\dagger \sinh t] + [a^\dagger \sinh t, a \cosh t] \\
&= \cosh t \sinh t - \sinh t \cosh t = 0.
\end{aligned} \tag{44}$$

(44) also implies that $[b^\dagger, b^\dagger] = 0$ if we take its complex conjugate. Now, we need to find some state $|\Omega'\rangle$ which is the vacuum for b. Let's assume that $|\Omega'\rangle = f(a^\dagger)|0\rangle$, where $|0\rangle$ is the vacuum state for a, since any function of a will just annihilate the vacuum. Then

$$\begin{aligned}
b|\Omega'\rangle &= (a\cosh t + a^\dagger \sinh t) f(a^\dagger)|0\rangle \\
&= ([a, f(a^\dagger)]\cosh t + a^\dagger f(a^\dagger)\sinh t)|0\rangle \\
&= \left(\frac{df}{da^\dagger}\cosh t + a^\dagger f(a^\dagger)\sinh t\right)|0\rangle .
\end{aligned} \tag{45}$$

We want the factor in front of $|0\rangle$ to disappear, which gives us a differential equation for f:

$$\begin{aligned}
\frac{df}{da^\dagger}\cosh t &= -a^\dagger f(a^\dagger)\sinh t \\
\frac{df}{f} &= -a^\dagger \tanh t \\
\ln f &= -\frac{1}{2}(a^\dagger)^2 \tanh t + C \\
f(a^\dagger) &= \exp\left[-\frac{1}{2}(a^\dagger)^2 \tanh t\right],
\end{aligned} \tag{46}$$

where we will absorb the constant into the normalization of the state. Thus, we have

$$\boxed{|\Omega'\rangle = \exp\left[-\frac{1}{2}(a^\dagger)^2 \tanh t\right]|0\rangle .} \tag{47}$$

2. (e)

This boils down to solving for t in the Bogolyubov transformation. In this case, we can have a different t_i for each set of raising and lowering operators.

$$\begin{aligned}
\omega_i' b_i^\dagger b_i &= \omega_i'(a_i^\dagger \cosh t_i + a_i \sinh t_i)(a_i \cosh t_i + a_i^\dagger \sinh t_i) \\
&= \omega_i' a_i^\dagger a_i \cosh^2 t_i + \omega_i'[a_i^2 + (a_i^\dagger)^2] \sinh t_i \cosh t_i + \omega_i' a_i a_i^\dagger \sinh^2 t_i \\
&= \omega_i' a_i^\dagger a_i (\cosh^2 t_i + \sinh^2 t_i) + \omega_i'[a_i^2 + (a_i^\dagger)^2] \sinh t_i \cosh t_i + \omega_i' \sinh^2 t_i \\
&= \omega_i' a_i^\dagger a_i \cosh 2t_i + \frac{\omega_i'}{2}[a_i^2 + (a_i^\dagger)^2] \sinh 2t_i + \omega_i' \sinh^2 t_i.
\end{aligned} \tag{48}$$

This implies

$$\omega_i = \omega_i' \cosh 2t_i, \tag{49}$$

$$\lambda_i = \omega_i' \sinh 2t_i. \tag{50}$$

Then

$$\begin{aligned}
\omega_i^2 - \lambda_i^2 &= \omega_i'^2(\cosh^2 2t_i - \sinh^2 2t_i) \\
&= \omega_i'^2
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
\omega_i' \sinh^2 t_i &= \frac{\omega_i'}{2}(\cosh 2t_i - 1) \\
&= \frac{\omega_i'}{2}\left(\frac{\omega_i}{\omega_i'} - 1\right) \\
&= \frac{1}{2}(\omega_i - \omega_i').
\end{aligned} \tag{52}$$

Finally, these imply

$$\boxed{\omega_i' = \sqrt{\omega_i^2 - \lambda_i^2},} \tag{53}$$

$$\boxed{E_0 = \sum_i \frac{1}{2}\left(\sqrt{\omega_i^2 - \lambda_i^2} - \omega_i\right).} \tag{54}$$