1. The Lorentz group in three spacetime dimensions is three dimensional because there is one way to rotate (in the x-y plane) and two ways to boost (in either the x or y directions). Alternatively, there are three generators of the group $SO(2,1)$.

   To construct the homomorphism in question, let's find the generators of the two groups. Once we have found these, we should be able to specify the mapping by exponentiating the generators of each group. Let's look at $SO(2,1)$ first. The defining representation of this group consists of matrices Λ for which

$$\Lambda^T \eta \Lambda = \eta, \tag{1}$$

   where

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

   Taking λ to be an infinitesimal generator of the Lorentz transformations gives us

$$(1 + i\lambda^T)\eta(1 + i\lambda) = \eta, \tag{3}$$

   which implies

$$\eta\lambda = -\lambda^T \eta. \tag{4}$$

   Writing out the components of λ gives us

$$\begin{gathered} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = - \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -a & -b & -c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & -d & -g \\ b & -e & -h \\ c & -f & -i \end{pmatrix}. \end{gathered} \tag{5}$$

   (5) implies that λ is of the form

$$\lambda = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix}. \tag{6}$$

   An good basis for the generators (i.e. the Lie algebra $so(2,1)$) is then

$$T = \left\{ K_x = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \right\}. \tag{7}$$

The first two generators correspond to boosts and the last corresponds to a rotation. Now, let's try to do the same for $SL(2,\mathbb{R})$. Because the determinant of these matrices is 1, we know that for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{8}$$

Expanding in terms of infinitesimal parameters near the identity gives

$$\begin{aligned} 1 &= \begin{pmatrix} 1+\delta a & \delta b \\ \delta c & 1+\delta d \end{pmatrix} \begin{pmatrix} 1+\delta d & -\delta b \\ -\delta c & 1+\delta a \end{pmatrix} \\ &= \begin{pmatrix} 1+\delta a+\delta d & 0 \\ 0 & 1+\delta a+\delta d \end{pmatrix}. \end{aligned} \tag{9}$$

Thus, we see that $\delta d = -\delta a$ is the only constraint on the generators. A basis for the generators which reproduces the commutation relations for the generators of $so(2,1)$ specified in (7) and satisfies the constraint from (9) is

$$T' = \frac{1}{2}\left\{-i\sigma_x, -i\sigma_z, \sigma_y\right\}, \tag{10}$$

so any $M \in SL(2,\mathbb{R}) = \exp\left(\eta_x\sigma_x + \eta_y\sigma_z + i\theta\sigma_y\right)$ for some choice of η_x,η_y, and θ. We then see that the homomorphism from $f : SL(2,\mathbb{R}) \to SO(2,1)$ should be given by

$$f(\exp[i\vec{n}\cdot\vec{T}']) = \exp[i\vec{n}\cdot\vec{T}], \tag{11}$$

where $\vec{T}$ and $\vec{T}'$ are vectors consisting of the basis of generators in (7) and (10) respectively. We can see that this homomorphism is 2-1 by noting that $\vec{n} = (0,0,0)$ and $\vec{n} = (0,0,2\pi)$ both give the same element in $SO(2,1)$, the identity, but because of the factor of $\frac{1}{2}$ in the definition of the generators T' give different elements of $SL(2,\mathbb{R})$, namely the identity and negative one times the identity. In fact, the previous statement is sufficient to guarantee that the homomorphism is 2-1, because the first isomorphism theorem tells us that

$$SO(2,1) = f[SL(2,(R))] \cong \frac{SL(2,\mathbb{R})}{\ker(f)} = \frac{SL(2,\mathbb{R})}{\mathbb{Z}_2}. \tag{12}$$

It is self dual, because

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right]^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}. \tag{13}$$

Finally, there is only one inequivalent spinor representation in 3D.

2. Srednicki 36.3

36.3) a) We have $(\chi_1^\dagger\bar\sigma^\mu\chi_2)(\chi_3^\dagger\bar\sigma_\mu\chi_4) = \bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot c c}\,\chi^\dagger_{1\dot a}\chi_{2a}\chi^\dagger_{3\dot c}\chi_{4c}$. Then we use $\bar\sigma^{\mu\dot a a}\bar\sigma_\mu^{\dot c c} = -2\varepsilon^{ac}\varepsilon^{\dot a\dot c}$ and $\chi^\dagger_{1\dot a}\chi_{2a}\chi^\dagger_{3\dot c}\chi_{4c} = -\chi^\dagger_{1\dot a}\chi^\dagger_{3\dot c}\chi_{2a}\chi_{4c}$ along with $\varepsilon^{ac}\chi_c = \chi^a$ and its dotted counterpart to get $(\chi_1^\dagger\bar\sigma^\mu\chi_2)(\chi_3^\dagger\bar\sigma_\mu\chi_4) = 2\chi^\dagger_{1\dot a}\chi_3^{\dagger\dot a}\chi_{2a}\chi_4^a = -2\chi^\dagger_{1\dot a}\chi_3^{\dagger\dot a}\chi_2^a\chi_{4a} = -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)$, which is eq. (36.58). Then we use $\chi_2\chi_4 = \chi_4\chi_2$, and go backwards throught these steps to get the right-hand side of eq. (36.59).

b) Using eqs. (36.7), (36.22), (36.45), and (36.60), we find $\overline\Psi_1\gamma^\mu P_{\rm L}\Psi_2 = \chi_1^\dagger\bar\sigma^\mu\chi_2$, $\overline\Psi_1 P_{\rm R}\Psi_3^{\rm C} = \chi_1^\dagger\chi_3^\dagger$, and $\overline\Psi{}_4^{\rm C}P_{\rm L}\Psi_2 = \chi_4\chi_2$, which yield eqs. (36.61–62) from eqs. (36.58–59).

c) In terms of Weyl fields, we have $\overline\Psi_1\gamma^\mu P_{\rm R}\Psi_2 = \xi_1\sigma^\mu\xi_2^\dagger = -\xi_2^\dagger\bar\sigma^\mu\xi_1 = -\overline\Psi{}_2^{\rm C}\gamma^\mu P_{\rm L}\Psi_1^{\rm C}$, $\overline\Psi_1 P_{\rm L}\Psi_2 = \xi_1\chi_2 = \chi_2\xi_1 = \overline\Psi{}_2^{\rm C}P_{\rm L}\Psi_1^{\rm C}$, and $\overline\Psi_1 P_{\rm R}\Psi_2 = \chi_1^\dagger\xi_2^\dagger = \xi_2^\dagger\chi_1^\dagger = \overline\Psi{}_2^{\rm C}P_{\rm R}\Psi_1^{\rm C}$.

3. Srednicki 36.4

36.4) a) This form for $T^{\mu\nu}$ is identical to eq. (22.29). The derivation is unchanged if the index a is replaced with the Lorentz index A.

b) For $\Lambda = 1+\delta\omega$, the Lorentz transformation $\varphi_A(x) \to L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x)$ becomes $\varphi_A(x) \to (\delta_A{}^B + \frac{i}{2}\delta\omega_{\nu\rho}(S^{\nu\rho})_A{}^B)(\varphi_B(x) - \delta\omega_{\nu\rho}x^\rho\partial^\nu\varphi_B(x))$, so that $\delta\varphi_A = \delta\omega_{\nu\rho}(-x^\rho\partial^\nu\varphi_A + \frac{i}{2}(S^{\nu\rho})_A{}^B)\varphi_B$. Also, $\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$ implies $\delta\mathcal{L} = -\delta\omega_\nu{}^\rho x_\rho\partial^\nu\mathcal{L} = \partial^\mu(-\delta\omega_{\nu\rho}g^{\mu\nu}x^\rho\mathcal{L})$; we then identify $K^\mu = -\delta\omega_{\nu\rho}g^{\mu\nu}x^\rho\mathcal{L}$. Using eq. (22.27), we then have

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\delta\varphi_A - K^\mu \\ &= \delta\omega_{\nu\rho}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}(-x^\rho\partial^\nu\varphi_A) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\tfrac{i}{2}(S^{\nu\rho})_A{}^B\varphi_B + g^{\mu\nu}x^\rho\mathcal{L}\right] \\ &= \delta\omega_{\nu\rho}\left[x^\rho T^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\,\tfrac{i}{2}(S^{\nu\rho})_A{}^B\varphi_B\right] \\ &= -\tfrac{1}{2}\delta\omega_{\nu\rho}\left[x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}(S^{\nu\rho})_A{}^B\varphi_B\right], \end{aligned} \tag{36.81}$$

and we identify the object in square brackets as

$$\mathcal{M}^{\mu\nu\rho} \equiv x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + B^{\mu\nu\rho}\,, \tag{36.82}$$

where

$$B^{\mu\nu\rho} \equiv -i\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}(S^{\nu\rho})_A{}^B\varphi_B\,. \tag{36.83}$$

c) Consider $\partial_\mu \mathcal{M}^{\mu\nu\rho}$; we have $\partial_\mu(x^\nu T^{\mu\rho}) = \delta_\mu{}^\nu T^{\mu\rho} + x^\nu \partial_\mu T^{\mu\nu} = T^{\nu\rho} + 0 = T^{\nu\rho}$, and so $0 = \partial_\mu \mathcal{M}^{\mu\nu\rho} = T^{\nu\rho} - T^{\rho\nu} + \partial_\mu B^{\mu\nu\rho}$.

d) We have $\Theta^{\mu\nu} \equiv T^{\mu\nu} + \frac{1}{2}\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$. Note that because (by definition) $S^{\mu\nu} = -S^{\nu\mu}$, eq. (36.83) implies $B^{\rho\mu\nu} = -B^{\rho\nu\mu}$. Note also that the last two terms in $\Theta^{\mu\nu}$ are symmetric on $\mu \leftrightarrow \nu$. Thus we have $\Theta^{\mu\nu} - \Theta^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \partial_\rho B^{\rho\mu\nu}$, which vanishes according to the result of part (c).

Next consider $\partial_\mu \Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) = \frac{1}{2}\partial_\mu\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu})$. Note that $B^{\rho\mu\nu} - B^{\mu\rho\nu} + B^{\nu\rho\mu}$ is antisymmetric on $\mu \leftrightarrow \rho$, and therefore vanishes when acted on by the symmetric derivative combination $\partial_\mu\partial_\rho$.

$\Theta^{0\nu} = T^{0\nu} + \frac{1}{2}\partial_\rho(B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) = T^{0\nu} + \frac{1}{2}\partial_i(B^{i0\nu} - B^{0i\nu} - B^{\nu i0})$. The integral over d^3x of $\frac{1}{2}\partial_i(\ldots)$ vanishes (assuming suitable boundary conditions at spatial infinity) because it is a total divergence. Therefore $P^\nu = \int d^3x\, T^{0\nu} = \int d^3x\, \Theta^{0\nu}$.

e) Recall from part (c) that $\partial_\mu(x^\nu \Theta^{\mu\rho}) = \Theta^{\nu\rho}$ if $\partial_\mu \Theta^{\mu\nu} = 0$. We have $\Xi^{\mu\nu\rho} \equiv x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}$, and so $\partial_\mu \Xi^{\mu\nu\rho} = \Theta^{\nu\rho} - \Theta^{\rho\nu} = 0$.

$$\begin{aligned}
\Xi^{\mu\nu\rho} &= x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + \tfrac{1}{2}x^\nu \partial_\sigma(B^{\sigma\mu\rho} - B^{\mu\sigma\rho} - B^{\rho\sigma\mu}) - \tfrac{1}{2}x^\rho \partial_\sigma(B^{\sigma\mu\nu} - B^{\mu\sigma\nu} - B^{\nu\sigma\mu}) \\
&= \mathcal{M}^{\mu\nu\rho} - B^{\mu\nu\rho} + \tfrac{1}{2}x^\nu \partial_\sigma(B^{\sigma\mu\rho} - B^{\mu\sigma\rho} - B^{\rho\sigma\mu}) - \tfrac{1}{2}x^\rho \partial_\sigma(B^{\sigma\mu\nu} - B^{\mu\sigma\nu} - B^{\nu\sigma\mu}) \,,
\end{aligned}$$

and so

$$\Xi^{0\nu\rho} = \mathcal{M}^{0\nu\rho} - B^{0\nu\rho} + \tfrac{1}{2}x^\nu \partial_i(B^{i0\rho} - B^{0i\rho} - B^{\rho i0}) + \tfrac{1}{2}x^\rho \partial_i(B^{i0\nu} - B^{0i\nu} - B^{\nu i0}) \,.$$

Now using $x^\nu \partial_i(\ldots) = \partial_i[x^\nu(\ldots)] - (\ldots)\partial_i x^\nu = \partial_i[x^\nu(\ldots)] - (\ldots)\delta_i{}^\nu$, we get

$$\begin{aligned}
\Xi^{0\nu\rho} &= \mathcal{M}^{0\nu\rho} - B^{0\nu\rho} - \tfrac{1}{2}(B^{\nu 0\rho} - B^{0\nu\rho} - B^{\rho\nu 0}) + \tfrac{1}{2}(B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0}) + \partial_i[\ldots] \\
&= \mathcal{M}^{0\nu\rho} - \tfrac{1}{2}(B^{0\nu\rho} + B^{0\rho\nu}) - \tfrac{1}{2}(B^{\nu 0\rho} + B^{\nu\rho 0}) + \tfrac{1}{2}(B^{\rho 0\nu} + B^{\rho\nu 0}) + \partial_i[\ldots] \\
&= \mathcal{M}^{0\nu\rho} + \partial_i[\ldots] \,.
\end{aligned}$$

Since the last term is a total divergence, $M^{\nu\rho} = \int d^3x\, \mathcal{M}^{0\nu\rho} = \int d^3x\, \Xi^{0\nu\rho}$.

(f) The improved energy-momentum tensor $\Theta^{\mu\nu}$ is given by

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) \,. \tag{14}$$

Therefore the key task is to compute $T^{\mu\nu}$ and $B^{\rho\mu\nu}$ for the given theories. For a left-handed Weyl field,

$$T^{\mu\nu} = g^{\mu\nu}\left[i\psi^\dagger\bar\sigma^\rho\partial_\rho\psi - \frac{1}{2}m\psi\psi - \frac{1}{2}m\psi^\dagger\psi^\dagger\right] - i\psi^\dagger\bar\sigma^\mu\partial^\nu\psi \,, \tag{15}$$

$$B^{\mu\nu\rho} = \frac{i}{4}\psi^\dagger\bar\sigma^\mu[\sigma^\nu\bar\sigma^\rho - \sigma^\rho\bar\sigma^\nu]_A{}^B\psi_B \,. \tag{16}$$

For a Dirac field,

$$T^{\mu\nu} = g^{\mu\nu}(i\bar\Psi\gamma^\rho\partial_\rho\Psi - m\bar\Psi\Psi) - i\bar\Psi\gamma^\mu\partial^\nu\Psi \,, \tag{17}$$

$$B^{\mu\nu\rho} = \bar\Psi^A\gamma^\mu(S^{\nu\rho})_A{}^B\Psi_B \,, \tag{18}$$

where $S^{\nu\rho}$ is given by

$$S^{\nu\rho} = \frac{i}{4}[\gamma^\nu, \gamma^\rho]\,. \tag{19}$$

Plugging $T^{\mu\nu}$ and $B^{\mu\nu\rho}$ into Eq. (14) will give the corresponding energy-momentum tensor $\Theta^{\mu\nu}$.