1.

We start by writing $\Delta(\mathbf{x})$ in terms of the Fourier transforms of the field:

$$\begin{aligned}\Delta(\mathbf{x}) &= \int \frac{\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^6(2\omega_{\vec{k}})(2\omega_{\vec{q}})}[a_{\vec{k}}e^{ikx} + a^\dagger_{\vec{k}}e^{-ikx}, a_{\vec{q}} + a^\dagger_{\vec{q}}] \\ &= \int \frac{\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^6(2\omega_{\vec{k}})(2\omega_{\vec{q}})}(2\pi)^3(2\omega_{\vec{q}})\delta^3(\vec{k}-\vec{q})(e^{ikx} - e^{-ikx}) \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3(2\omega_{\vec{k}})}(e^{ikx} - e^{-ikx}).\end{aligned} \tag{1}$$

I'll proceed by using polar coordinates. Note that $\omega_{\vec{k}} = |\vec{k}|$. The integral then becomes

$$\begin{aligned}\Delta(\mathbf{x}) &= \int \frac{r^2\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\cos\theta}{(2\pi)^3 2r}\left[e^{-irt}e^{irx\cos\theta} - c.c.\right] \\ &= \frac{1}{2(2\pi)^2}\int r\,\mathrm{d}r\left[e^{-irt}\int_{-1}^{1} d\cos\theta e^{irx\cos\theta} - c.c.\right] \\ &= \frac{1}{2(2\pi)^2}\int r\,\mathrm{d}r\left[\frac{e^{-irt}}{irx}(e^{irx} - e^{-irx}) - c.c.\right] \\ &= \frac{1}{2ix(2\pi)^2}\int_0^\infty \mathrm{d}r\left[e^{ir(x-t)} - e^{-ir(x+t)} + e^{-ir(x-t)} - e^{ir(x+t)}\right].\end{aligned} \tag{2}$$

Rearranging the limits of integration gives us

$$\begin{aligned}&\int_0^\infty \mathrm{d}r\left[e^{ir(x-t)} + e^{-ir(x-t)} - e^{-ir(x+t)} - e^{ir(x+t)}\right] \\ &\quad= \int_0^\infty \mathrm{d}re^{ir(x-t)} - \int_\infty^0 \mathrm{d}re^{-ir(x-t)} - \int_0^\infty \mathrm{d}re^{-ir(x+t)} + \int_\infty^0 \mathrm{d}re^{ir(x+t)} \\ &\quad= \int_0^\infty \mathrm{d}re^{ir(x-t)} + \int_{-\infty}^0 \mathrm{d}re^{ir(x-t)} - \int_0^\infty \mathrm{d}re^{-ir(x+t)} - \int_{-\infty}^0 \mathrm{d}re^{-ir(x+t)} \\ &\quad= 2\pi[\delta(x-t) - \delta(x+t)]\end{aligned} \tag{3}$$

This implies

$$\Delta(\mathbf{x}) = \frac{\delta(x-t) - \delta(x+t)}{4\pi ix}. \tag{4}$$

As it stands, this formula doesn't look very Lorentz invariant. We can put it in a nicer form by noticing that

$$\begin{aligned}\delta(x^2 - t^2) &= \frac{1}{2|x|}[\delta(x-t) + \delta(x+t)] \\ &= \operatorname{sgn}(t)\left[\frac{\delta(x-t)}{2x} - \frac{\delta(x+t)}{2x}\right].\end{aligned} \tag{5}$$

if we imagining that we are integrating over t. The second line is true because the delta function constrains the x in the second term to be negative whenever it is nonzero (since t is positive). With this in mind, we can see that

$$\boxed{\Delta(\mathbf{x}) = \operatorname{sgn}(t)\frac{\delta(\mathbf{x}^2)}{2\pi i}.} \tag{6}$$

2.

The lowering operators will annihilate the vacuum on the right and lowering operators will do so on the left. This means we have

$$\begin{aligned}\langle 0|\, \phi(\mathbf{x})\phi(0)\, |0\rangle &= \int \frac{\mathrm{d}^3k\, \mathrm{d}^3q}{(2\pi)^6(2\omega_{\vec{k}})(2\omega_{\vec{q}})} \langle 0|\, a_{\vec{k}} e^{ikx} a_{\vec{q}}^\dagger\, |0\rangle \\ &= \int \frac{\mathrm{d}^3k\, \mathrm{d}^3q}{(2\pi)^6(2\omega_{\vec{k}})(2\omega_{\vec{q}})} e^{ikx} \langle 0|\, [a_{\vec{k}}, a_{\vec{q}}^\dagger]\, |0\rangle \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3(2\omega_{\vec{k}})} e^{ikx}.\end{aligned} \tag{7}$$

Let's try to use the hint from the problem to evaluate this (we could also have used the hint in the last problem, if we were careful).

$$\begin{aligned}\langle 0|\, \phi(\mathbf{x})\phi(0)\, |0\rangle &= \frac{1}{(2\pi)^3}\int \mathrm{d}^4k\theta(k^0)\delta(-\mathbf{k}^2)e^{ikx} \\ &= \frac{1}{(2\pi)^3}\int \mathrm{d}k^0\, \mathrm{d}k^1\theta(k^0)e^{ikx}\int \mathrm{d}k^2\, \mathrm{d}k^3\delta(-\mathbf{k}^2)\end{aligned} \tag{8}$$

Now we go to polar coordinates to find the integral over the delta function:

$$\begin{aligned}\int \mathrm{d}k^2\,\mathrm{d}k^3\delta(-\mathbf{k}^2) &= 2\pi \int r\,\mathrm{d}r\delta(k_0^2 - k_1^2 - r^2)\\ &= 2\pi \int r\,\mathrm{d}r\frac{\delta(r-\sqrt{k_0^2-k_1^2})}{2\sqrt{k_0^2-k_1^2}}\theta(k_0^2-k_1^2)\\ &= \pi\theta(k_0^2-k_1^2).\end{aligned} \tag{9}$$

Then our equation becomes

$$\begin{aligned}\langle 0|\,\phi(\mathbf{x})\phi(0)\,|0\rangle &= \frac{\pi}{(2\pi)^3}\int \mathrm{d}k^0\,\mathrm{d}k^1\theta(k^0)\theta(k_0^2-k_1^2)e^{ikx}\\ &= \frac{\pi}{(2\pi)^3}\int_{k^0\geq|k^1|}\mathrm{d}k^0\,\mathrm{d}k^1 e^{ikx}.\end{aligned} \tag{10}$$

If we define coordinates $k_\pm = k^0 \pm k^1$, we see $k^0 = (k_+ + k_-)/2$ and $k^1 = (k_+ - k_-)/2$, which implies the Jacobian of the transformation is $1/4$. Furthermore, the region we integrate over is just the whole quadrant with $k_\pm \geq 0$. Finally, we see $k\cdot x = -k_+x_- - k_-x_+$, where $x_\pm = (x^0 \pm x^1)/2$. Then we have

$$\langle 0|\,\phi(\mathbf{x})\phi(0)\,|0\rangle = \frac{\pi}{4(2\pi)^3}\int \mathrm{d}k_+ e^{-ik_+x_-}\theta(k_+)\int \mathrm{d}k_- e^{-ik_-x_+}\theta(k_-). \tag{11}$$

This looks likes a product of Fourier transforms of the step function. Looking this up gives us

$$\int \frac{\mathrm{d}k}{2\pi}e^{-ikx}\theta(k) = \frac{1}{2}\delta(x) - \frac{1}{2\pi i x}. \tag{12}$$

Thus, we have

$$\begin{aligned}\langle 0|\,\phi(\mathbf{x})\phi(0)\,|0\rangle &= \frac{1}{16}\left[\delta(x_-) - \frac{1}{i\pi x_-}\right]\left[\delta(x_+) - \frac{1}{i\pi x_+}\right]\\ &= \frac{-1}{16}\left[\frac{1}{\pi^2 x_+x_-} + \frac{\delta(x_-)}{i\pi x_+} + \frac{\delta(x_+)}{i\pi x_-} - \delta(x_-)\delta(x_+)\right]\end{aligned} \tag{13}$$

The middle two terms match the result from the last problem. The last term should be interpreted in the following manner: if we integrate over x_+ and x_-, there is only a contribution if $x_+ = x_- = 0$ is included in the integration range. We can therefore write it as

$$\delta(x_-)\delta(x_+) = \delta(x_0)\delta(x_1), \tag{14}$$

since $x_+ = x_- = 0$ implies $x_0 = x_1 = 0$ and (by construction) the Jacobian from changing variables in the measure is exactly compensated by factors from the delta functions. Finally, x_1 is actually equivalent to $|\vec{x}|$ here (we rotated to place $\vec{x}$ along the x_1 axis), so we should write

$$\delta(x_-)\delta(x_+) = \delta(x_0)\delta(|\vec{x}|). \tag{15}$$

While this doesn't look particularly Lorentz invariant, it actually is — it simply says that there is a contribution to an integral only if the point $x_0 = \vec{x} = 0$ is included in the integration region, and this statement does not depend on our frame. Putting this all together, and remembering that as we've defined things $x_+x_- = -\mathbf{x}^2/4$, we find

$$\boxed{\langle 0|\,\phi(\mathbf{x})\phi(0)\,|0\rangle = \frac{1}{4\pi^2\mathbf{x}^2} + \mathrm{sgn}(t)\frac{\delta(\mathbf{x}^2)}{4\pi i} + \frac{\delta(t)\delta(|\vec{x}|)}{16}.} \tag{16}$$

3.

$$\begin{aligned} i\partial_0\phi &= [H, \phi] \\ &= \left[\int \mathrm{d}^3x(pp^\dagger + \partial_i\phi^\dagger\partial_i\phi + m^2\phi^d\phi), \phi(y)\right] \\ &= \int \mathrm{d}^3x[p, \phi(y)]p^\dagger \\ &= -ip^\dagger. \end{aligned} \tag{17}$$

This implies

$$\begin{aligned} i\partial_0^2\phi &= -i\partial_0 p^\dagger \\ &= [p^\dagger, H] \\ &= \int \mathrm{d}^3x[p^\dagger(y), \partial_i\phi^\dagger]\partial_i\phi + m^2[p^\dagger(y), \phi^\dagger]\phi \\ &= (-i)(-\nabla^2)\phi + (-i)m^2\phi, \end{aligned} \tag{18}$$

so we have

$$\boxed{(\partial_0^2 - \nabla^2 + m^2)\phi = 0.} \tag{19}$$

4. (a)

Let's consider the properties of a near-identity infinitesimal rotation by writing $R = 1 + i\delta R$. Then we have

$$\begin{aligned} R^T R &= 1 \\ (1 + i\delta R^T)(1 + i\delta R) &= 1 \\ \delta R^T + \delta R &= 0. \end{aligned} \tag{20}$$

Then the generators of rotations (the matrices δR) must be antisymmetric. This implies that they can be parameterized as

$$\delta R^{ab} = \epsilon^{abc}\beta^c, \tag{21}$$

which means we can write out the infinitesimal transformation of the fields as

$$\boxed{\delta\phi^a = \epsilon^{abc}\phi^b\beta^c.} \tag{22}$$

To deduce the conserved currents, we see how the action changes under the above transformation if we treat the parameters β as functions of spacetime:

$$\begin{aligned} \delta S &= -\int \mathrm{d}^4x \partial_\mu \delta\phi^a \partial^\mu \phi^a \\ &= -\int \mathrm{d}^4x \epsilon^{abc}(\beta^c \partial_\mu \phi^b \partial^\mu \phi^a + \partial_\mu \beta^c \phi^b \partial^\mu \phi^a) \\ &= \int \mathrm{d}^4x \beta^c \partial_\mu(\epsilon^{abc}\phi^b\partial^\mu\phi^a). \end{aligned} \tag{23}$$

Now, if the fields follow a classical path (that is, if they satisfy the equations of motion), the variation of the action must vanish even under the circumstances where the β are (infinitesimal) arbitrary functions of spacetime. This implies that on the equations of motion,

$$\partial_\mu(\epsilon^{abc}\phi^b\partial^\mu\phi^a) = 0. \tag{24}$$

That means our conserved currents are given by

$$\boxed{J^{a\mu} = \epsilon^{abc}\phi^b\partial^\mu\phi^c.} \tag{25}$$

The above argument is just Noether's theorem. See section 7.3 of Weinberg I for a good summary.

4. (b)

We see that the Qs are given by

$$Q^a = \int \mathrm{d}^3x \epsilon^{abc} \phi^b \dot{\phi}^c. \tag{26}$$

Then we have

$$\begin{aligned}
[Q^a, Q^b] &= \int \mathrm{d}^3x \, \mathrm{d}^3y \epsilon^{acd} \epsilon^{bkl} [\phi^c \dot{\phi}^d, \phi^k \dot{\phi}^l] \\
&= \int \mathrm{d}^3x \, \mathrm{d}^3y \epsilon^{acd} \epsilon^{bkl} (\phi^c [\dot{\phi}^d, \phi^k] \dot{\phi}^l + [\phi^c, \phi^k] \dot{\phi}^d \dot{\phi}^l + \phi^k \phi^c [\dot{\phi}^d, \dot{\phi}^l] + \phi^k [\phi^c, \dot{\phi}^l] \dot{\phi}^d) \\
&= \epsilon^{acd} \epsilon^{bkl} \int \mathrm{d}^3x \, \mathrm{d}^3y [-i\delta^3(x-y) \delta^{dk} \phi^c \dot{\phi}^l + i\delta^3(x-y) \delta^{cl} \phi^k \dot{\phi}^d] \\
&= -i\epsilon^{acd} \epsilon^{bdl} \int \mathrm{d}^3x \phi^c \dot{\phi}^l + i\epsilon^{acd} \epsilon^{bkc} \int \mathrm{d}^3x \phi^k \dot{\phi}^d \\
&= -i(\delta^{al} \delta^{cb} - \delta^{ab} \delta^{cl}) \int \mathrm{d}^3x \phi^c \dot{\phi}^l + i(\delta^{db} \delta^{ak} - \delta^{dk} \delta^{ab}) \int \mathrm{d}^3x \phi^k \dot{\phi}^d \\
&= i(\delta^{ak} \delta^{bd} - \delta^{ad} \delta^{bl}) \int \mathrm{d}^3x \phi^k \dot{\phi}^d \\
&= i\epsilon^{cab} \epsilon^{ckd} \int \mathrm{d}^3x \phi^k \dot{\phi}^d \\
&= \boxed{i\epsilon^{abc} Q^c.}
\end{aligned} \tag{27}$$