1. We are dealing with a two-component real spinor field in 2+1-dimension.

   (a) In order to write down the Lagrangian, we need to find a representation of the γ matrices. In 2+1-dimension, we can take $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ and $\gamma^2 = i\sigma_3$, where σ_i are the Pauli matrices. We can check that

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\,, \tag{1}$$

   where

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{2}$$

   In this representation, all γ matrices are purely imaginary. If we write down the Dirac Lagrangian for this field Φ,

$$\mathcal{L} = \frac{i}{2}\bar{\Phi}\gamma^\mu\partial_\mu\Phi - \frac{1}{2}m\bar{\Phi}\Phi\,, \tag{3}$$

   the equation of motion that follows from this Lagrangian (i.e. Dirac equation) is

$$(-i\gamma^\mu\partial_\mu + m)\Phi = 0\,, \tag{4}$$

   which is purely real. It is then consistent to require the spinor field Φ to be real, which justifies our choices for the γ matrices.

   (b) CPT symmetry is guaranteed by the CPT theorem. That is, the Lagrangian is invariant under a simultaneous CPT transformation. Now let us discuss discrete transformation.

   Charge conjugation Because the spinor field is real-valued, this spinor field is a Majorana spinor, therefore the charge conjugation symmetry is built in. Let us show this precisely by solving for the Dirac equation. Considering a specific plane-wave solution of the form

$$\Phi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx}\,, \tag{5}$$

   where $u(\mathbf{p})$ and $v(\mathbf{p})$ are two-component constant spinors. Since Φ is real, $\Phi^*(x) = \Phi(x)$. This gives

$$v(\mathbf{p}) = u^*(\mathbf{p})\,. \tag{6}$$

   Plugging Eq. (5) into Eq. (4), we get

$$(\not{p} + m)u(\mathbf{p})e^{ipx} + (-\not{p} + m)u^*(\mathbf{p})e^{-ipx} = 0\,. \tag{7}$$

   Thus we require

$$\begin{aligned} (\not{p} + m)u(\mathbf{p}) &= 0\,, \\ (-\not{p} + m)u^*(\mathbf{p}) &= 0\,. \end{aligned} \tag{8}$$

For $m \neq 0$, we can go to the rest frame, $\mathbf{p} = 0$. We then have $\not{p} = \gamma^0 p_0$, and Eqs. (8) are now easy to solve. We obtain

$$(m \cdot \mathbb{1}_{2\times 2} + p_0 \sigma_2) u(\mathbf{p}) = 0\,, \tag{9}$$

a nontrivial solution exists for $\det(m \cdot \mathbb{1}_{2\times 2} + p_0\sigma_2) = 0$, which gives $p_0 = |m|$. Therefore the solution is given by

$$u(\mathbf{0}) = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \qquad m > 0 \tag{10}$$

$$u(\mathbf{0}) = \begin{pmatrix} \text{-i} \\ 1 \end{pmatrix}. \qquad m < 0 \tag{11}$$

Now that the spinors corresponding to an arbitrary three-momentum $\mathbf{p}$ can be found by applying an appropriate boost. Given m, since we have only one solution for $u(\mathbf{0})$, we should also have only one solution for a fixed momentum. Now the Fourier expansion of the field Φ should look like

$$\Phi(x) = \int d^3p \left[a(\mathbf{p}) u(\mathbf{p}) e^{ipx} + a^\dagger(\mathbf{p}) u^*(\mathbf{p}) e^{-ipx} \right]. \tag{12}$$

Notice that the two terms in the above expansion are obviously conjugate to each other, which is how the expression should give a real Φ. Upon quantization, it is then clear that for each momuntum p, there is exactly one creation operator $\hat{a}^\dagger$ and one annilation operator $\hat{a}$. Therefore the field describes particles which are their own anti-particles.

Parity transformation For nonzero mass, we can go to the rest frame $\mathbf{p} = 0$, and compute the eigenvalues of the spin matrix,

$$S_z = \frac{i}{4}[\gamma^1, \gamma^2] = -\frac{1}{2}\sigma_2\,. \tag{13}$$

Unlike in 3+1-dimension, the Dirac equation now describes one single polarization state for each momentum . Acting the spin operator on the solutions to the Dirac equation gives

$$S_z u(\mathbf{0}) = \frac{1}{2}\frac{|m|}{m} u(\mathbf{0})\,. \tag{14}$$

Therefore this polarization state has angular momentum $+1/2$ when $m > 0$, and $-1/2$ when $m < 0$. Since parity transformation reverses the angular momentum, P symmetry is broken. Only when $m = 0$, P symmetry is restored.

Another way to see why P symmetry is broken for $m \neq 0$ is to study the real Dirac equation under parity transformation directly. The real Dirac equation is given by

$$\{-i\sigma_2\partial_0 + \sigma_1\partial_1 + \sigma_3\partial_2 + m\}\Phi_p(t, x_1, x_2) = 0\,. \tag{15}$$

Under the parity transformation $x_1 \to -x_1$ and $x_2 \to x_2$[1], the Dirac equation becomes

$$\{-i\sigma_2\partial_0 - \sigma_1\partial_1 + \sigma_3\partial_2 + m\}\Phi(t, -x_1, x_2) = 0\,. \tag{16}$$

Now we ask whether this equation can be brought to its original form Eq. (15) by the mere applications of unitary operators. Such an operator, however, does not exist and the wave function $\Phi(t, x_1, x_2)$ and $\Phi(t, -x_1, x_2)$ cannot be transformed into each other. A unitary operator U made of σ_i's that restores the signs in Eq. (17) must have the properties $\{U, \sigma_1\} = 0$, $[U, \sigma_2] = 0$ and $[U, \sigma_3] = 0$. These requirements are impossible to meet in the algebra spanned by all pauli matrices and their products.

In the absence of a mass term ($m = 0$), however, it turns out such transformation exists. The relation

$$\{-i\sigma_2\partial_0 - \sigma_1\partial_1 + \sigma_3\partial_2\}\Phi(t, -x_1, x_2) = 0\,. \tag{17}$$

can be transformed by applying σ_1 from the left

$$\{-i\sigma_2\partial_0 - \sigma_1\partial_1 + \sigma_3\partial_2\}\sigma_1\Phi(t, -x_1, x_2) = 0\,. \tag{18}$$

which is exactly the Dirac equation for $m = 0$. That is, the massless Dirac equation for the two-component real spinor is invariant under $x_1 \to -x_1$ and the solution is related by

$$\Phi(t, x_1, x_2) = \eta\sigma_1\Phi(t, -x_1, x_2)\,, \tag{19}$$

where η is a phase factor.

Time reversal We have already shown that for the two component real spinor field, the real Dirac equation only describes a single polarization, and this state has momentum $+1/2$ for $m > 0$ and $-1/2$ for $m < 0$. Since T symmetry also reverses the angular momentum in nonzero mass case, the T symmetry is broken. Only PT symmety exists for $m \neq 0$. T symmetry is restored for $m = 0$.

One can also show this by studying the time reversal of the real Dirac equation, and constructing the unitary operator that transforms $\Phi(-t, x_1, x_2)$ to $\Phi(t, x_1, x_2)$. This operation should leave the Dirac equation invariant. The whole procedure will be completely analogous to the discussion from Eq. (15) to (19). It turns out such operator only exists for the zero mass limit.

2. Srednicki 36.5

   (a) The transformation matrix must be orthogonal to preserve the mass term, hence the symmetry is $O(N)$.

   (b) A Majorana field is equivalent to a Weyl field, hence the symmetry is $U(N)$.

[1] Note in odd dimensions, the parity transformation is usually defined as flipping all space coordinates except the last one. This is because in odd dimension, the space part is even-dimentional and flipping all of the coordinates is equivalent to a Lorentz transformation.

(c) Combining the results of parts (a) and (b), the symmetry is $O(N)$.

(d) A Dirac field is equivalent to two Weyl fields, hence the symmetry is $U(2N)$.

(e) The symmetry is $U(N)$. A Dirac fermion is equivalent to two Weyl fermions with a mass term. But the mass term is different from that in Eq. 36.76. The correct Lagrangian is given in Eq. 37.12, because the mass term is now $m\chi\xi + C.C$, there is no $O(2N)$ symmetry. But still, the Dirac Lagrangian is invariant under the unitary transformation of the Dirac spinor, therefore there is $U(N)$ symmetry.