1. (a)

Take everything to be in 1+1 dimenstions. Inserting the resolution of the identity and remembering what $\langle p|q\rangle$ is gives us

$$\begin{aligned} K(q',q;T) &= \langle q'| e^{-iHT} \int \mathrm{d}p \, |p\rangle \langle p|q\rangle \\ &= \int \mathrm{d}p \, \langle q'| e^{-ip^2T/2m} |p\rangle \langle p|q\rangle \\ &= \int \frac{\mathrm{d}p}{2\pi} e^{-ip^2T/2m} e^{ipq'} e^{-ipq} \\ &= \int \frac{\mathrm{d}p}{2\pi} \exp\left\{ \frac{-iT}{2m} \left[p^2 + \frac{2m(q-q')}{T} p \right] \right\} \\ &= \exp\left[\frac{im(q'-q)^2}{2T} \right] \int \frac{\mathrm{d}p}{2\pi} \exp\left\{ \frac{-iT}{2m} \left[p + \frac{m(q-q')}{T} \right]^2 \right\}. \end{aligned} \tag{1}$$

The remaining integral is Gaussian, and we get

$$\begin{aligned} K(q',q;T) &= \frac{1}{2\pi} \sqrt{\frac{\pi}{iT/2m}} \exp\left[\frac{im(q'-q)^2}{2T} \right] \\ &= \boxed{\sqrt{\frac{m}{2\pi iT}} \exp\left[\frac{im(q'-q)^2}{2T} \right].} \end{aligned} \tag{2}$$

1. (b)

First, let's prove that the propagator and the 2-point Green's function are equal:

$$\begin{aligned} G(q',q;T) &\equiv \langle 0| \Psi(T,q') \Psi^\dagger(0,q) |0\rangle \\ &= \langle 0| e^{-iHT} \Psi(0,q') e^{iHT} \Psi^\dagger(0,q) |0\rangle \\ &= \langle 0| \Psi(0,q') e^{iHT} \Psi^\dagger(0,q) |0\rangle . \end{aligned} \tag{3}$$

The last line comes from the fact that $H|0\rangle = 0 \rightarrow e^{-iHT}|0\rangle = |0\rangle$. Then we have

$$\boxed{G(q',q;T) = \langle q'| e^{iHT} |q\rangle = K(q',q;T),} \tag{4}$$

since $\Psi^\dagger(0,q)|0\rangle = |q\rangle$. To confirm this in the case $V=0$, expand in Fourier series:

$$\begin{aligned}
G(q',q;T) &= \langle 0| \int \frac{\mathrm{d}p}{2\pi} a_k e^{-iE_kT+ikq'} \int \frac{\mathrm{d}p}{2\pi} a_p^\dagger e^{-ipq} |0\rangle \\
&= \int \frac{\mathrm{d}p\,\mathrm{d}k}{(2\pi)^2} e^{-iE_kT+ikq'} e^{-ipq} \langle 0| [a_k, a_p^\dagger] |0\rangle \\
&= \int \frac{\mathrm{d}p\,\mathrm{d}k}{(2\pi)^2} e^{-iE_kT+ikq'} e^{-ipq} (2\pi)\delta(p-k) \\
&= \int \frac{\mathrm{d}p}{2\pi} e^{-ip^2T/2m} e^{ipq'} e^{-ipq}.
\end{aligned} \tag{5}$$

This is (literally) the same integral we already evaluated in (1), so the solution is the same:

$$\boxed{G(q',q;T) = \sqrt{\frac{m}{2\pi i T}} \exp\left[\frac{im(q'-q)^2}{2T}\right] = K(q',q;T).} \tag{6}$$

1. (c)

We could evaluate the Gaussian integral directly, but let's just see what happens for N=1. In this case, there is no integral, and $\epsilon = T$. Then we have

$$\begin{aligned}
K_1(q',q;T) &= F(\epsilon)^1 \exp\left[\frac{i\epsilon m}{2\epsilon^2}(q'-q)\right] \\
&= F(\epsilon) \exp\left[\frac{im(q'-q)^2}{2T}\right].
\end{aligned} \tag{7}$$

Apparently when there is no potential the path integral will give the correct answer with any choice of subdivision of the path. Then (7) will match (6) if

$$\boxed{F(\epsilon) = \sqrt{\frac{m}{2\pi i \epsilon}}.} \tag{8}$$

1. (d)

It is possible to do this using matrices, but let's just start by seeing what happens if we integrate over a particular q_i:

$$\begin{aligned} &K(q,q';T) \\ &\quad= \lim_{N\to\infty} F(\epsilon)^N \int \mathrm{d}q_1 \ldots \mathrm{d}q_{N-1} \exp\left\{ i\epsilon \sum_{i=1}^{N-1} \left[\frac{m(q_{i+1}-q_i)^2}{2\epsilon^2} + fq_i \right] \right\} \\ &\qquad = \cdots \int \mathrm{d}q_i \exp \frac{im}{2\epsilon} \left[(q_{i+1}-q_i)^2 + \frac{2\epsilon^2}{m} f q_i + (q_i - q_{i-1})^2 \right] \ldots \end{aligned} \tag{9}$$

Expanding and factoring out the terms which don't depend on q_i gives us

$$\begin{aligned} K(q,q';T) &= \ldots \exp\left[\frac{im}{2\epsilon}(q_{i+1}^2 + q_{i-1}^2)\right] \\ &\quad \times \int \mathrm{d}q_i \exp\left\{ \frac{im}{2\epsilon}\left[-2q_{i+1}q_i + q_i^2 + \frac{2\epsilon^2}{m} f q_i + q_i^2 - 2q_i q_{i-1} \right] \right\} \ldots \\ &= \ldots \exp\left[\frac{im}{2\epsilon}(q_{i+1}^2 + q_{i-1}^2)\right] \\ &\quad \times \int \mathrm{d}q_i \exp\left\{ \frac{im}{\epsilon}\left[q_i^2 - \left(q_{i+1} + q_{i-1} - \frac{\epsilon^2}{m} f \right) q_i \right] \right\} \ldots \end{aligned} \tag{10}$$

We can complete the square for q_i:

$$\begin{aligned} &q_i^2 - \left(q_{i+1} + q_{i-1} - \frac{\epsilon^2}{m} f \right) q_i \\ &\qquad = \left[q_i - \frac{1}{2}\left(q_{i+1} + q_{i-1} - \frac{\epsilon^2}{m} f \right) \right]^2 - \frac{1}{4}\left(q_{i+1} + q_{i-1} - \frac{\epsilon^2}{m} f \right)^2. \end{aligned} \tag{11}$$

Then we have

$$\begin{aligned} &K(q,q';T) \\ &\qquad = \ldots \exp\left\{ \frac{im}{4\epsilon}\left[2q_{i+1}^2 + 2q_{i-1}^2 - \left(q_{i+1} + q_{i-1} - \frac{\epsilon^2}{m} f \right)^2 \right] \right\} \\ &\qquad\qquad \times \int \mathrm{d}q_i' \exp\left(\frac{im}{\epsilon} q_i'^2 \right) \ldots \end{aligned} \tag{12}$$

$$= \dots \exp\left[\frac{im}{4\epsilon}(q_{i+1}-q_{i-i})^2 + \frac{i\epsilon}{2}f(q_{i+1}+q_{i-1}) - \frac{i\epsilon^3}{4m}f^2\right]\sqrt{\frac{\pi}{-im\epsilon}}\dots$$

$$= \dots \sqrt{\frac{\pi i\epsilon}{m}}\exp\left(-\frac{i\epsilon^3}{4m}f^2\right) \times \exp\left[\frac{im}{4\epsilon}(q_{i+1}-q_{i-i})^2 + \frac{i\epsilon}{2}f(q_{i+1}+q_{i-1})\right]\dots \quad (13)$$

Now, take N to be even and do the integrals over q_i for odd i. Then we are left with the following:

$$K(q,q';T) = \lim_{N\to\infty} F(\epsilon)^N \sqrt{\frac{\pi i\epsilon}{m}}^{N/2} \times \exp\left(-\frac{i\epsilon^3}{4m}f^2\right)^{N/2} \exp\left[\frac{i\epsilon}{2}f(q_N+q_0)\right] \times \int \{\mathrm{d}q\} \exp\left\{\sum_j \left[\frac{im(q_{j+1}-q_j)^2}{4\epsilon} - 2i\epsilon f q_j\right]\right\}. \quad (14)$$

$\{\mathrm{d}q\}$ represents the remaining variables that have not yet been integrated over, and j should be taken to run over only these variables. The remaining integral is identical to our original integral, but with $\epsilon \to 2\epsilon$. This makes sense, because we are effectively doubling the time step between points. So we see what will happen if we iterate this process: we will halve the number of steps each time and produce a factor out front. Let's evaluate the prefactor P out front, taking $N = 2^M$:

$$P = F(\epsilon)^N \prod_{j=0}^{M-1} \sqrt{\frac{2^j\pi i\epsilon}{m}}^{N/2^{j+1}} \prod_{k=0}^{M-1} \exp\left(-\frac{i2^{3k}\epsilon^3}{4m}f^2\right)^{N/2^{k+1}} \times \prod_{q=0}^{M-1} \exp\left[\frac{i2^q\epsilon}{2}f(q_N+q_0)\right]. \quad (15)$$

First, note that the first two factors are identical to the free case, since they just come from performing the Gaussian integrals. This implies that they must give

$$F(\epsilon)^N \prod_{j=0}^{M-1} \sqrt{\frac{2^j\pi i\epsilon}{m}}^{N/2^{j+1}} \to \sqrt{\frac{m}{2\pi iT}} \quad (16)$$

as N goes to infinity. Now let's deal with the edge term. We have

$$\begin{aligned}\prod_{q=0}^{M-1} \exp\left[\frac{i2^q\epsilon}{2} f(q_N+q_0)\right] &= \exp\left[\frac{i\epsilon}{2} f(q_N+q_0) \sum_{q=0}^{M-1} 2^q\right] \\ &= \exp\left[\frac{i\epsilon}{2} f(q_N+q_0)\frac{2^M-1}{2-1}\right] \\ &\to \exp\left[\frac{iN\epsilon}{2} f(q_N+q_0)\right] \\ &= \exp\left[\frac{iT}{2} f(q_N+q_0)\right].\end{aligned} \tag{17}$$

The remaining factor is

$$\begin{aligned}\prod_{k=0}^{M-1} \exp\left(-\frac{i2^{3k}\epsilon^3}{4m} f^2\right)^{N/2^{k+1}} &= \exp\left(-\frac{iN\epsilon^3}{4m} f^2 \sum_{k=0}^{M-1} \frac{2^{3k}}{2^{k+1}}\right) \\ &= \exp\left(-\frac{iN\epsilon^3}{4m} f^2 \sum_{k=0}^{M-1} 2^{2k-1}\right).\end{aligned} \tag{18}$$

But

$$\begin{aligned}\sum_{k=0}^{M-1} 2^{2k-1} &= \frac{1}{2}\sum_{k=0}^{M-1} 4^k \\ &= \frac{4^M-1}{2(4-1)} \\ &= \frac{1}{6}(2^{2M}-1) \\ &= \frac{1}{6}(N^2-1),\end{aligned} \tag{19}$$

so

$$\begin{aligned}\prod_{k=0}^{M-1} \exp\left(-\frac{i2^{3k}\epsilon^3}{4m} f^2\right)^{N/2^{k+1}} &\to \exp\left(-\frac{iN^3\epsilon^3 f^2}{24m}\right) \\ &= \exp\left(-\frac{iT^3 f^2}{24m}\right)\end{aligned} \tag{20}$$

Then the prefactor is

$$P \to \exp\left[\frac{iT}{2} f(q_N + q_0)\right] \exp\left(-\frac{iT^3 f^2}{24m}\right), \tag{21}$$

and the final expression for the path integral should be

$$\begin{aligned} &K(q, q'; T) \\ &\quad \to \sqrt{\frac{m}{2\pi i T}} \exp\left[\frac{iT}{2} f(q' + q)\right] \exp\left(-\frac{iT^3 f^2}{24m}\right) \exp\left[\frac{im(q' - q)^2}{4(2^{M-1}\epsilon)}\right] \\ &\qquad\qquad = \boxed{\sqrt{\frac{m}{2\pi i T}} \exp\left[\frac{im}{2T}(q' - q)^2 + \frac{iT}{2} f(q' + q) - \frac{iT^3 f^2}{24m}\right].} \end{aligned} \tag{22}$$