1. Srednicki 48.5

48.5) Let $g \equiv c_1 G_F f_\pi$; the vertex factor is then $(ig)(ik_\mu)\gamma^\mu(1-\gamma_5) = -g\not{k}(1-\gamma_5)$, where k is the four-momentum of the pion. Thus we have $i\mathcal{T} = -g\overline{u}_1\not{k}(1-\gamma_5)v_2$, where p_1 is the muon momentum and p_2 is the antineutrino momentum. We now use $\not{k} = \not{p}_1 + \not{p}_2$, $\not{p}_2(1-\gamma_5) = (1+\gamma_5)\not{p}_2$, $\overline{u}_1\not{p}_1 = -m_\mu\overline{u}_1$, and $\not{p}_2 v_2 = 0$ to get $\mathcal{T} = -igm_\mu\overline{u}_1(1-\gamma_5)v_2$. Then $\overline{\mathcal{T}} = +igm_\mu\overline{v}_2(1+\gamma_5)u_1$, and $|\mathcal{T}|^2 = g^2m_\mu^2\,\mathrm{Tr}[u_1\overline{u}_1(1-\gamma_5)v_2\overline{v}_2(1+\gamma_5)]$. Summing over final spins yields

$$
\begin{aligned}
\langle|\mathcal{T}|^2\rangle &= g^2m_\mu^2\mathrm{Tr}[(-\not{p}_1+m_\mu)(1-\gamma_5)(-\not{p}_2)(1+\gamma_5)] \\
&= g^2m_\mu^2\,\mathrm{Tr}[(-\not{p}_1+m_\mu)(-\not{p}_2)(1+\gamma_5)(1+\gamma_5)] \\
&= 2g^2m_\mu^2\,\mathrm{Tr}[(-\not{p}_1+m_\mu)(-\not{p}_2)(1+\gamma_5)] \\
&= 2g^2m_\mu^2\,\mathrm{Tr}[\not{p}_1\not{p}_2] \\
&= 2g^2m_\mu^2\,(-4p_1p_2) \\
&= 4g^2m_\mu^2\,[-(p_1+p_2)^2 + p_1^2 + p_2^2] \\
&= 4g^2m_\mu^2\,(-k^2 + p_1^2 + p_2^2) \\
&= 4g^2m_\mu^2\,(m_\pi^2 - m_\mu^2 + 0)\ . \qquad (48.60)
\end{aligned}
$$

We then have $\Gamma = \langle|\mathcal{T}|^2\rangle|\mathbf{p}_1|/8\pi m_\pi^2$, and $|\mathbf{p}_1| = (m_\pi^2 - m_\mu^2)/2m_\pi^2$, so

$$\Gamma = \frac{g^2m_\mu^2}{4\pi m_\pi^3}(m_\pi^2 - m_\mu^2)^2\ . \qquad (48.61)$$

Using $\Gamma = \hbar c/c\tau = (1.973\times10^{-14}\,\mathrm{GeV\,cm})/(2.998\times10^{10}\,\mathrm{cm/s})(2.603\times10^{-8}\,\mathrm{s}) = 2.528\times10^{-17}\,\mathrm{GeV}$, we find $g = 1.058\times10^{-6}\,\mathrm{GeV}$, and so $f_\pi = 93.14$; after including electromagnetic loop corrections, the result drops slightly to $f_\pi = 92.4$.

2. Srednicki 52.3

(a) Since $dg/d\ln\mu = b_0 g^3/16\pi^2$ and $d\lambda/d\ln\mu = (c_0 g^4 + c_1\lambda g^2 + c_2\lambda^2)/16\pi^2$, for $\rho \equiv \lambda/g^2$ we have (by the chain rule)

$$\begin{aligned} \frac{d\rho}{d\ln\mu} &= \frac{g^2}{16\pi^2}\left(c_0 + (c_1 - 2b_0)\rho + c_2\rho^2\right) \qquad (1) \\ &= \frac{g^2}{16\pi^2} c_2(\rho - \rho_+^*)(\rho - \rho_-^*)\,, \end{aligned}$$

where $\rho_\pm^* = [2b_0 - c_1 \pm \sqrt{(c_1 - 2b_0)^2 - 4c_0c_2}]/2c_2$. Working with g and ρ is better because the beta function for ρ is now separable. For our case, $b_0 = 5, c_0 = -48, c_1 = 8, c_2 = 3$.

(b) $d\rho/d\ln\mu = 0$ gives two solutions, $\rho_\pm^* = (1 \pm \sqrt{145})/3 = 4.32$ and -3.68.

(c) Since g is small, we can treat it as approximately constant. For $\rho = 0$, β_ρ is negative, and so ρ increases as μ decreases, and approaches ρ_+^* from below; ρ decreases as μ increases, and approaches ρ_-^* from above.

(d) When $\rho = 5$, $\beta_\rho > 0$, ρ flows to ρ_+^* in the IR (low energy limit), and runs off to infinity in the UV (high energy limit).

(e) When $\rho = -5$, $\beta_\rho > 0$, ρ flows to ρ_-^* in the UV, and runs off to negative infinity in the IR.

(f) We have $d\rho/dg = \beta_\rho/\beta_g = (c_2/b_0)(\rho - \rho_+^*)(\rho - \rho_-^*)/g^2$. This can be separated and integrated to get

$$\int \frac{d\rho}{(\rho - \rho_+^*)(\rho - \rho_-^*)} = \frac{c_2}{b_0}\int \frac{dg}{g}\,, \qquad (2)$$

$$\frac{1}{\rho_+^* - \rho_-^*}\ln\left|\frac{\rho - \rho_+^*}{\rho - \rho_-^*}\right| = \frac{c_2}{b_0}\ln|g/g_0|\,, \qquad (3)$$

which yields the claimed result with $\nu = b_0/[c_2(\rho_+^* - \rho_-^*)] = 0.208$. The RG flow diagram is shown in Figure. 1.

(g) As energy increases, RG flow runs away from ρ_+^*, thus ρ_+^* is a IR fixed point; RG flow runs to ρ_-^*, thus ρ_-^* is a UV fixed point.

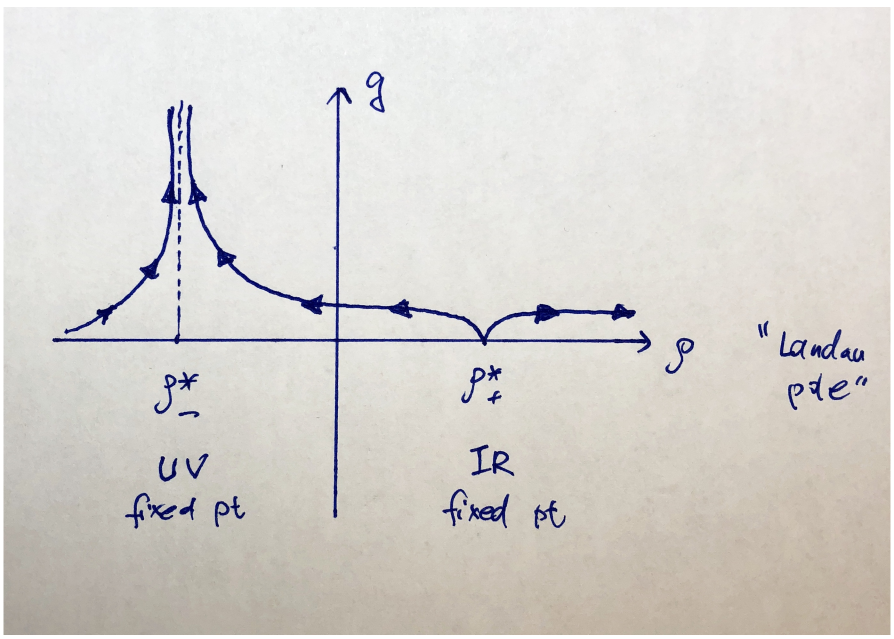

Figure 1: RG flow diagram in $\rho - g$ plane. Arrows are directed towards where energy is increasing.