1. (a) Gauge transformations take $A_\mu \rightarrow A_\mu + \partial_\mu f$, where $f$ is a scalar function of spacetime. This means

$$F_{\mu\nu} \rightarrow \partial_\mu A_\nu + \partial_\mu \partial_\nu f - \partial_\nu A_\mu - \partial_\nu \partial_\mu f = F_{\mu\nu}. \quad (1)$$

Thus, the field strength is gauge invariant by itself, and we only need to worry about the Chern-Simons term. Then

$$\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \rightarrow \epsilon^{\mu\nu\rho} (A_\mu + \partial_\mu f) \partial_\nu (A_\rho + \partial_\rho f)$$

$$= \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} A_\mu \partial_\rho \partial_\nu f + \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho + \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho \quad (2)$$

Also,

$$\int d^3 x \epsilon^{\mu\nu\rho} \partial_\mu f \partial_\nu A_\rho = - \int d^3 x \epsilon^{\mu\nu\rho} \partial_\nu f \partial_\rho A_\mu = 0 \quad (3)$$

after integration by parts, so we have

$$\int d^3 x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \rightarrow \int d^3 x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad (4)$$

and we see that the action is gauge invariant. Now let’s find the EoM for $A$ using the Euler-Lagrange equation. First, we have

$$\frac{\partial L}{\partial A_\mu} = \frac{1}{2} k \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$$

$$= \frac{1}{4} k \epsilon^{\mu\nu\rho} (\partial_\nu A_\rho - \partial_\rho A_\nu)$$

$$= \frac{1}{4} k \epsilon^{\mu\nu\rho} F_{\nu\rho}. \quad (5)$$

Next, using

$$\frac{\partial F_{\rho\sigma}}{\partial (\partial_\nu A_\mu)} = \delta_\rho^{\nu} \delta_\sigma^{\mu} - \delta_\sigma^{\nu} \delta_\rho^{\mu}, \quad (6)$$

we have

$$\frac{\partial L}{\partial (\partial_\nu A_\mu)} = - \frac{1}{2} \frac{\partial F_{\rho\sigma}}{\partial (\partial_\nu A_\mu)} F_{\rho\sigma} + \frac{1}{2} k \epsilon^{\rho\mu\nu} A_\rho$$

$$= - \frac{1}{2} (F_{\mu\nu} - F_{\nu\mu}) + \frac{1}{2} k \epsilon^{\rho\mu\nu} A_\rho$$

$$= F_{\mu\nu} + \frac{1}{2} k \epsilon^{\rho\mu\nu} A_\rho, \quad (7)$$

which implies

$$\partial_\nu \frac{\partial L}{\partial (\partial_\nu A_\mu)} = \partial_\nu F_{\mu\nu} + \frac{1}{2} k \epsilon^{\rho\mu\nu} \partial_\nu A_\rho$$

$$= \partial_\nu F_{\mu\nu} - \frac{1}{2} k \epsilon^{\rho\mu\nu} \partial_\nu A_\rho$$

$$= \partial_\nu F_{\mu\nu} - \frac{1}{4} k \epsilon^{\rho\mu\nu} F_{\nu\rho}. \quad (8)$$
Putting the pieces together, we have that the equations of motion for the gauge field are

\[
\begin{align*}
\frac{1}{4} k \epsilon^{\mu\nu\rho} F_{\nu\rho} &= \partial_{\nu} F^{\mu\nu} - \frac{1}{4} k \epsilon^{\mu\nu\rho} F_{\nu\rho} \\
\frac{1}{2} k \epsilon^{\mu\nu\rho} F_{\nu\rho} &= \partial_{\nu} F^{\mu\nu} \\
\epsilon_{\mu}^{\alpha\beta} \epsilon^{\mu\nu\rho} F_{\nu\rho} &= \frac{2}{k} \epsilon_{\mu}^{\alpha\beta} \partial_{\nu} F^{\mu\nu} \\
(\delta^{\alpha\nu} \delta^{\beta\rho} - \delta^{\alpha\rho} \delta^{\beta\nu}) F_{\nu\rho} &= \frac{2}{k} \epsilon_{\mu}^{\alpha\beta} \partial_{\nu} F^{\mu\nu} \\
F_{\alpha\beta} &= \frac{1}{k} \epsilon_{\mu}^{\alpha\beta} \partial_{\nu} F^{\mu\nu}.
\end{align*}
\]

This implies

\[
F_{\alpha\beta} = \frac{1}{k} \epsilon_{\mu}^{\alpha\beta} \partial_{\nu} \left( \frac{1}{k} \epsilon^{\mu\nu\rho} \partial_{\sigma} F^{\rho\sigma} \right) = \frac{1}{k^2} \partial_{\nu} \partial_{\sigma} (\delta^{\alpha\nu} \delta^{\beta\rho} - \delta^{\alpha\rho} \delta^{\beta\nu}) F^{\rho\sigma} = \frac{1}{k^2} \left( \partial_{\nu} \partial_{\sigma} F^{\beta\sigma} - \partial_{\rho} \partial_{\sigma} F^{\alpha\sigma} \right) = \frac{1}{k^2} \left[ \partial_{\nu} \partial_{\sigma} (\partial^{\beta} A^{\sigma} - \partial^{\sigma} A^{\beta}) - \partial^{\beta} \partial_{\sigma} (\partial^{\alpha} A^{\sigma} - \partial^{\sigma} A^{\alpha}) \right] = \frac{1}{k^2} (\partial^{\beta} \partial_{\sigma} A^{\beta} + \partial^{\beta} \partial_{\sigma} A^{\alpha})
\]

The final line of (10) is equivalent to

\[
(k^2 + \partial^2) F_{\mu\nu} = 0,
\]

so we see that \( F_{\mu\nu} \) satisfies the Klein-Gordon equation with mass \( m^2 = k^2 \). Since we are in 2 + 1 dimensions, these massive particles have 1 polarization, which comes from the fact that \( A_0 \) is nondynamical and we can eliminate one more component through gauge transformations.

(b) The Gauss law constraint comes from the Euler-Lagrange equation for \( A_0 \). We have

\[
\frac{\partial L}{\partial A_0} = \frac{1}{2} k \epsilon^{0\nu\rho} \partial_{\nu} A_{\rho} = \frac{1}{2} k B.
\]

Furthermore,

\[
\frac{\partial L}{\partial (\partial_\nu A_0)} = \partial_\nu F^{0\nu} - \frac{1}{4} k \epsilon^{0\nu\rho} F_{\nu\rho} = \partial_\nu F^{0\nu} - \frac{1}{2} k B = \nabla \cdot \vec{E} - \frac{1}{2} k B
\]
Then the analog of the Gauss law constraint is
\[
\nabla \cdot \vec{E} = kB.
\]
(14)

Now let's find the momentum conjugate to \( A_i \). We have from (7)
\[
\frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = F^{i0} + \frac{1}{2} k \epsilon^{ijkl} A_j,
\]
(15)
which implies
\[
\dot{p}^i = -E^i + \frac{1}{2} k \epsilon^{ijkl} A_j.
\]
(16)

Before we find the Hamiltonian, let's rewrite the Lagrangian a bit. First, notice that
\[
F_{\mu\nu} F^{\mu\nu} = F_{i0} F^{i0} + F_{ij} F^{ij},
\]
(17)
but
\[
F_{i0} F^{i0} = - (\partial_i A_0 - \partial_0 A_i)^2 = -E^2
\]
(18)
and
\[
F_{ij} F^{ij} = F_{12} F^{12} + F_{21} F^{21} = 2 (\partial_1 A_2 - \partial_2 A_1)^2 = 2B^2.
\]
(19)

Then we have
\[
\mathcal{L} = -\frac{1}{2} (-E^2 + B^2) + \frac{1}{2} k (\epsilon^{0ij} A_0 \partial_i A_j + \epsilon^{j0i} A_j \partial_0 A_i + \epsilon^{ij0} A_i \partial_j A_0).
\]
(20)

The equation for the Hamiltonian is
\[
\mathcal{H} = \dot{p}^i \partial_0 A_i - \mathcal{L}.
\]
(21)

We have
\[
\dot{p}^i \partial_0 A_i = -E^i \partial_0 A_i + \frac{1}{2} k \epsilon^{ij0} A_j \partial_0 A_i,
\]
(22)
so (21) implies
\[
\mathcal{H} = -E^i \partial_0 A_i + \frac{1}{2} (-E^2 + B^2) - \frac{1}{2} k (\epsilon^{0ij} A_0 \partial_i A_j + \epsilon^{ij0} A_i \partial_j A_0).
\]
(23)

Integration by parts gives us
\[
\epsilon^{ij0} A_i \partial_j A_0 \rightarrow - A_0 \epsilon^{ij} \partial_j A_i = A_0 B.
\]
(24)
Thus, we have
\[
\mathcal{H} = -E^i \partial_0 A_i + \frac{1}{2} (-E^2 + B^2) - A_0 kB.
\]
(25)
We can then write

\[ \mathcal{H} = -E^i(\partial_i A_0 - E_i) + \frac{1}{2}(-E^2 + B^2) - A_0 k B \]

\[ = E^2 + \frac{1}{2}(-E^2 + B^2) - E^i \partial_i A_0 - A_0 k B \]

\[ \rightarrow \frac{1}{2}(E^2 + B^2) + A_0(\partial_i E^i - kB) \]

We integrated by parts in the last step. Thus, we see that \( A_0 \) acts as a Lagrange multiplier which enforces (14), and assuming the Gauss’s law constraint (i.e. plugging back in the equation of motion for \( A_0 \)) gives us

\[ \mathcal{H} = \frac{1}{2}(E^2 + B^2). \]

I think this is actually what the problem wanted. Now, we can use (16) to write

\[ E^i = \frac{1}{2} k \varepsilon^{ij} A_j - p^i, \]

which implies

\[ E^2 = \frac{1}{4} k^2 \varepsilon^{ij} \varepsilon_{ik} A_j A^k - k \varepsilon^{ij} p_i A_j + p^2 \]

\[ = \frac{1}{4} k^2 A^2 - k \varepsilon^{ij} p_i A_j + p^2, \]

so the action can be written

\[ S = \int d^3 x \left[ p^i \partial_0 A_i - \frac{1}{2} \left( p^2 - k \varepsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) - A_0 \left( \partial_i E^i - kB \right) \right] \]

\[ = \int d^3 x \left[ p^i \partial_0 A_i - \frac{1}{2} \left( p^2 - k \varepsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) - A_0 \left( \frac{1}{2} k B - \partial_i p^i - kB \right) \right] \]

\[ = \int d^3 x \left[ p^i \partial_0 A_i - \frac{1}{2} \left( p^2 - k \varepsilon^{ij} p_i A_j + \frac{1}{4} k^2 A^2 + B^2 \right) + A_0 \left( \partial_i p^i + \frac{1}{2} k B \right) \right]. \]

(d) The Poisson brackets are given by

\[ \{ E_i(x), B(y) \} = \{-p_i + \frac{1}{2} \varepsilon_{ij} A_j, \varepsilon_{kl} \partial_k A_l \} \]

\[ = -\varepsilon_{kl} \partial_k \{ p_i, A_l \} + \frac{1}{2} \varepsilon_{ij} \varepsilon_{kl} \partial_k \{ A_j, A_l \} \]

\[ = -\varepsilon_{kl} \partial_k \delta^2(x - y) + \frac{1}{2} \varepsilon_{ij} \varepsilon_{kl} \partial_k(0) \]

\[ = \varepsilon_{ik} \partial_k \delta^2(x - y). \]
(e) We can compute the propagator for the photon by writing down the Lagrangian in momentum space and inverting the coefficient of $A^2$. Because this theory is gauge invariant (see part (a)), we will need to gauge fix. We can do this by adding a gauge fixing term to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \frac{1}{2\xi} (\partial_\mu A^\nu)^2. \quad (32)$$

Then the Lagrangian in momentum space looks like

$$\mathcal{L} = \frac{1}{2} \left[ A_\mu \left( p^2 g^{\mu\nu} - p^\mu p^\nu - \frac{1}{\xi} p^\mu p^\nu + i k \epsilon^{\mu\nu\rho} p_\rho \right) A_\nu \right]. \quad (33)$$

Defining

$$[\Delta^{-1}]^{\mu\nu}(p^2) = i \left[ p^2 g^{\mu\nu} - \left( 1 + \frac{1}{\xi} \right) p^\mu p^\nu + i k \epsilon^{\mu\nu\rho} p_\rho \right], \quad (34)$$

the most general form for $\Delta_{\mu\nu}(p^2)$ allowed by Lorentz invariance is

$$\Delta_{\mu\nu}(p^2) = -i(a g_{\mu\nu} + b p_\mu p_\nu + c \epsilon_{\mu\nu\rho} p^\rho), \quad (35)$$

Where $a$, $b$, and $c$ are functions of $p^2$ and $k$. Note that this is possible because $k$ has mass dimension one. Then we have

$$\Delta_{\mu\nu}[\Delta^{-1}]^{\nu\lambda} = (a g_{\mu\nu} + b p_\mu p_\nu + c \epsilon_{\mu\nu\rho} p^\rho) \times \left[ p^2 g^{\rho\lambda} - \left( 1 + \frac{1}{\xi} \right) p^\rho p^\lambda + i k \epsilon^{\rho\lambda\mu} p_\mu \right]$$

$$= a p^2 \delta^\lambda_\mu + \left[ b - a \frac{a}{\xi} - b p^2 \left( 1 + \frac{1}{\xi} \right) \right] p_\mu p^\lambda + (cp^2 + iak) \epsilon^{\lambda\mu}_\rho p_\rho + ic k \epsilon_{\mu\nu\rho} \epsilon^{\nu\lambda\tau} p^\mu p^\tau \quad (36)$$

But

$$\epsilon_{\mu\nu\rho} \epsilon^{\nu\lambda\sigma} p^\rho p_\sigma = p_\mu p^\lambda - p^2 \delta_\mu^\lambda, \quad (37)$$

so we have

$$\Delta_{\mu\nu}[\Delta^{-1}]^{\nu\lambda} = (a - i c k) p^2 \delta^\lambda_\mu + \left[ b - a \frac{a}{\xi} - b p^2 \left( 1 + \frac{1}{\xi} \right) + i c k \right] p_\mu p^\lambda$$

$$+ (cp^2 + iak) \epsilon^{\lambda\mu}_\rho p_\rho. \quad (38)$$

To start, this implies

$$c p^2 + i a k = 0 \quad (39)$$

$$c = -\frac{iak}{p^2}. \quad (39)$$

Furthermore, we must have

$$a - i c k = \frac{1}{p^2}, \quad (40)$$
so we get

\[
 a - a \frac{k^2}{p^2} = \frac{1}{p^2} \\
 a = \frac{1}{p^2 - k^2}.
\]  

(41)

This also implies from (39) that

\[
 c = \frac{ik}{p^2(k^2 - p^2)}.
\]  

(42)

In the Lorenz gauge, $\xi \to 0$, so we should only worry about the terms with $\frac{1}{\xi}$. Then we have

\[
 \frac{a}{\xi} + bp^2 \frac{1}{\xi} = 0 \\
 b = -\frac{a}{p^2} \\
 = \frac{1}{p^2(k^2 - p^2)}.
\]  

(43)

Altogether, the expression for the propagator is

\[
 \Delta_{\mu\nu}(p^2) = \frac{-i}{p^2 + k^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{ik \epsilon_{\mu\nu\rho\sigma} p^\rho}{p^2} \right).
\]  

(44)