1 Consider the theory of a gauge field $A_{\mu}$ in three-dimensional space-time with a Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}k\epsilon^{\mu\nu\rho}A_{\mu}\partial_{\nu}A_{\rho}.$$ 

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The second term in the Lagrangian is called the Chern-Simons term, and the theory is called Chern-Simons-Maxwell theory.

(a) Show that the action $S = \int L d^3x$ is gauge-invariant. Derive the equations of motion for the field $A$. Show that they imply that each component of the tensor $F_{\mu\nu}$ satisfies the massive Klein-Gordon equation for some mass $m$. Express $m$ in terms of $k$. How many polarization states to these ”massive waves” have?

(bc) In three dimensions the analog of the electric field $E_i = -\partial_0 A_i + \partial_i A_0$ is a vector, but the analog of the magnetic field $B = \epsilon_{ij}\partial_j A_i$ is a scalar. As in four dimensions, $A_0$ should be thought of as a Lagrange multiplier whose equation of motion gives a constraint (the analog of the Gauss law) and does not have a conjugate momentum, while the fields $A_i$, $i = 1, 2$, do. What is the analog of the Gauss law constraint $\partial_i E_i = 0$ in this theory? Find the momenta conjugate to $A_i$. Write the Hamiltonian in terms of $B$ and the momenta conjugate to $A_i$. Write the action in Hamiltonian form (i.e. in a form where the momenta and $A_i$ are regarded as independent and the action is linear in time derivatives).

(d) Compute the Poisson brackets of $E_i$ and $B$.

(e) Compute the propagator for $A$ in the Lorenz gauge $\partial^\mu A_{\mu} = 0$. 

Week 6 (due Feb. 14)