Week 1 (due Oct. 5)

Each (sub)problem is worth 10 pts.

1. Let A be an operator acting in the Hilbert space of a single particle. Consider a system of N identical particles labeled by $\alpha = 1, \ldots, N$ and the corresponding operators A_α each of which depends only on the coordinates and momenta of the α^{th} particle. Consider the following operator acting on the Hilbert space of N particles (which may be bosons or fermions)

$$A^{(N)} = \sum_{\alpha=1}^{N} A_\alpha.$$

Using the method of second quantization as explained in class, we can represent this operator as follows:

$$A^{(N)} = \sum_{i,j} a_i^\dagger a_j \langle i|A|j\rangle$$

Here $|i\rangle$ is an arbitrary orthonormal basis of states in the one-particle Hilbert space, a_i and $a_j^\dagger$ are the corresponding annihilation and creation operators, and summation extends over all basis states.

The current density operator in an N-particle system is defined to be

$$\mathbf{j}_\alpha(\mathbf{x}) = \frac{1}{2m} \sum_{\alpha=1}^{N} \left(\mathbf{p}_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha) + \delta(\mathbf{x} - \mathbf{x}_\alpha)\mathbf{p}_\alpha\right).$$

Here $\mathbf{p}_\alpha$ is the momentum operator for the α^{th} particle. Express the current density operator in terms of the creation-annihilation operators in the momentum basis (i.e. choosing momentum eigenstates as the basis) and in terms of field operators $\Psi(x)$ (i.e. choosing coordinate eigenstates as the basis).

2. (a) Consider a state in the bosonic Fock space which has the form

$$\exp\left(\sum_i \lambda_i a_i^\dagger\right)|0\rangle,$$

where λ_i are some complex numbers. What conditions should be imposed on λ_i to ensure that the state is normalizable? Determine the average number of particles in this state and the standard deviation.

(b) Consider modified bosonic creation and annihilation operators

$$b_i = a_i - \lambda_i, \quad b_i^\dagger = a_i^\dagger - \lambda_i^*.$$

Show that they satisfy the same commutation relations as $a_i, a_i^\dagger$ and show that the state in part (a) is the vacuum state for $b_i, b_i^\dagger$.

(c) Consider a Hamiltonian in the bosonic Fock space

$$H = \sum_i \left(\omega_i a_i^\dagger a_i + \beta_i a_i + \beta_i^* a_i^\dagger \right),$$

where ω_i are positive numbers and β_i are complex numbers. Show that this Hamiltonian does not conserve the particle number. Defining new creation and annihilation operators as above and choosing λ_i appropriately, show that the Hamiltonian in fact describes a collection of noninteracting bosonic particles, but with a modified ground state (i.e. the state with the lowest possible energy is not the vacuum state $|0\rangle$ for a_i). Determine the ground-state energy and the energies of one-particle excitations with respect to this ground state.

(d) Given bosonic creation-annihilation operators a and $a^\dagger$ and a real number t, consider their linear combinations

$$b = a \cosh t + a^\dagger \sinh t, \quad b^\dagger = a \sinh t + a^\dagger \cosh t.$$

Show that these operators satisfy the same commutation relations as $a, a^\dagger$. Determine the vacuum state with respect to the operators $b, b^\dagger$. (N.B. Such states are called squeezed states in the literature.)

(e) Consider the following Hamiltonian in the bosonic Fock space:

$$H = \sum_i \left(\omega_i a_i^\dagger a_i + \frac{1}{2}\lambda_i a_i a_i + \frac{1}{2}\lambda_i a_i^\dagger a_i^\dagger \right).$$

Here ω_i, λ_i are real numbers, and $\omega_i > 0$. Show that this Hamiltonian does not conserve the particle number. By defining suitable linear combinations $b_i, b_i^\dagger$ as in part (d), show that one can diagonalize this Hamiltonian. Show that in fact it describes a system of noninteracting bosons and determine the energy of the ground state and the energies of one-particle excitations.