Week 2 (due Oct. 12)

Each (sub)problem is worth 10 pts.

1. The creation/annihilation operators for the free scalar field are implicitly defined by the formula

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx} \right).$$

Express a_k and $a_k^\dagger$ in terms of $\phi(x)$ and $p(x) = \partial_0\phi(x)$ at fixed time $t = x^0$. Compute their commutators using equal-time commutation relations

$$[\phi(0,\mathbf{x}), p(0,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \tag{1}$$

$$[\phi(0,\mathbf{x}), \phi(0,\mathbf{y})] = 0, \tag{2}$$

$$[p(0,\mathbf{x}), p(0,\mathbf{y})] = 0. \tag{3}$$

2. It is natural to define the operator of spatial momentum for the free scalar field by a Fock-space expression

$$\mathbf{P} = \int \frac{d^3k}{(2\pi)^3}\, \mathbf{k}\, a_k^\dagger a_k.$$

(a) Show that $\mathbf{P}$ satisfies

$$[P^j, \phi(x)] = -i\partial_j\phi(x), \quad j = 1,2,3.$$

(b) Express $\mathbf{P}$ in terms of $\phi(x)$ and $p(x) = \partial_0\phi(x)$.

3. The Hamiltonian for the free scalar is given by

$$H = \frac{1}{2}\int d^3x \left(p^2 + \partial_i\phi\partial_i\phi + m^2\phi^2 \right).$$

Using equal-time commutators for $p(x)$ and $\phi(x)$, show that the Heisenberg equations of motion

$$i\partial_0\phi(x) = [\phi(x), H], \quad i\partial_0 p(x) = [p(x), H]$$

are equivalent to the Klein-Gordon equation for $\phi(x)$.

4. (a) Consider the free scalar field with $m = 0$. Compute $D_c(x) = [\phi(x), \phi(0)]$ for both time-like and space-like x. What can you say about the support of $D_c(x)$?

(b) The same as (a), but in a world with only one spatial dimension (that is, for a free massless scalar in two-dimensional space-time).