Week 3 (due Oct. 19)

Each (sub)problem is worth 10 pts.

1. (a) Consider the real scalar field with zero mass. Show that the following transformation is a symmetry of the action:

$$x' = \lambda x, \quad \phi'(x') = \lambda^{-1}\phi(x), \quad \lambda \neq 0.$$

This transformation is called a dilatation. Derive the conserved current corresponding to this symmetry and check by an explicit computation that it is conserved if the equations of motion are satisfied.

(b) Consider a massless scalar in n-dimensional space-time with the action

$$S = \frac{1}{2}\int d^n x \; \partial_\mu \phi \partial^\mu \phi.$$

Consider the transformation $x' = \lambda x$, where λ is a nonzero real number. Find the transformation law for $\phi(x)$ which makes the action invariant and compute the corresponding conserved current. Further, add the following term to the action:

$$\Delta S = \int d^n x \; V(\phi(x)),$$

where $V(\phi)$ is a function. What is the most general form of the function $V(\phi)$ compatible with the dilatation invariance? Compute the dilatation current for such $V(\phi)$.

(c) Consider the dilatation current computed in part (a). Let us call it D^μ. Using canonical commutation relations, show that the quantity

$$\int d^3 x \; D^0(x)$$

generates infinitesimal dilatations. That is, compute its equal-time commutator with $\phi(y)$ and show it is equal to $-i$ times the infinitesimal change in $\phi(y)$ under dilatations (the latter quantity was called $\delta\phi(x)$ in class).

(d) Show that if $T^\mu_\mu = 0$ on-shell, then the current $G^\mu = x^\nu T^\mu_\nu$ is conserved. If this is true, one says that the stress-energy tensor T^μ_ν is traceless. Show that the canonical stress-energy tensor for the massless scalar does not satisfy this property in 4d space-time, but does satisfy it in 2d space-time (i.e. for $n = 2$). Thus, for the massless scalar in two-dimensional space-time, there is a conserved current G^μ. What is its relation to the dilatation current D^μ?

2. (a) Consider a field theory with three real scalar fields $\phi^a(x), a = 1, 2, 3,$ and a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a(x)\partial^\mu \phi^a(x) - V(\phi^a\phi^a).$$

Here summation over repeating indices a is assumed, and V is an arbitrary function. This Lagrangian is obviously invariant with respect to orthogonal transformations of the fields ϕ^a:

$$\phi^a(x) \mapsto R^a_b \phi^b(x),$$

where R^a_b is a constant orthogonal 3×3 matrix. The rotation group in three dimensional space has dimension three, so we expect to get three conserved currents. Show that infintesimal transformations for $\phi^a(x)$ can be put into the form

$$\Delta^a \phi(x) = \epsilon^{abc}\phi^b(x)\beta^c,$$

where $\beta^c, c = 1, 2, 3$ parametrize an infintesimal rotation, and ϵ^{abc} is a completely anti-symmetric tensor uniquely defined by the condition $\epsilon^{123} = 1$. Deduce the conserved currents corresponding to this symmetry.

(b) Let the currents found in part (a) be called $J^{a,\mu}, a = 1, 2, 3$. The corresponding charges are

$$Q^a = \int d^3x J^{a,0}(x).$$

Compute the commutator of Q^a and Q^b using canonical commutation relations for ϕ^a and their time derivatives. Show that Q^a form a Lie algebra isomorphic to the Lie algebra of the rotation group (i.e. show that they obey the same commutation relations as components of the angular momentum operator in quantum mechanics). Compute also equal-time commutators of Q^a with $\phi^b(x)$ and show that Q^a generate infinitesimal rotations of the fields $\phi^a(x)$. That is, show that

$$i[\beta^c Q^c, \phi^a(x)] = \Delta^a \phi(x).$$