Week 3

Reading: Srednicki sec. 22, pp. 132-135.

1. Consider the theory of a complex scalar field ϕ. Let $j^\mu = i(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi)$. This is a conserved current, and the corresponding conserved charge is

$$Q = \int j^0 d^3x.$$

Express Q in terms of creation and annihilation operators $a, a^\dagger, b, b^\dagger$ and show that Q is the number of particles minus the number of antiparticles.

2. The real scalar field has action

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right).$$

It is natural to define the operator of spatial momentum for the free real scalar field by a Fock-space expression

$$\mathbf{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \mathbf{k}\, a_k^\dagger a_k.$$

(a) Show that $\mathbf{P}$ satisfies

$$[P^j, \phi(x)] = i\partial_j \phi(x), \quad j = 1, 2, 3.$$

(b) Express $\mathbf{P}$ in terms of $\phi(x)$ and $p(x) = \partial_0 \phi(x)$.

3. (a) Consider the real scalar field with $m = 0$. Show that the following transformation is a symmetry of the action:

$$\tilde{\phi}(x) = \lambda^{-1} \phi(\lambda^{-1} x), \quad \lambda \neq 0.$$

This transformation is called a dilatation. An alternative way to write this transformation is to let $\tilde{x} = \lambda x$ and write

$$\tilde{\phi}(\tilde{x}) = \lambda^{-1} \phi(x).$$

From this viewpoint, it is clear that dilatation symmetry originates from a geometric transformation $x \to \tilde{x} = \lambda x$.

Derive the conserved current corresponding to this symmetry and check by an explicit computation that it is conserved if the equations of motion are satisfied.

(b) Consider a massless scalar in n-dimensional space-time with the action

$$S_0 = -\frac{1}{2}\int d^n x\ \partial_\mu \phi \partial^\mu \phi.$$

Consider a transformation $x \longrightarrow \tilde{x} = \lambda x$, where λ is a nonzero real number. Find the transformation law for $\phi(x)$ which makes the action invariant and compute the corresponding conserved current. Further, add the following term to the action:

$$S_1 = -\int d^n x\ V(\phi(x)),$$

where $V(\phi)$ is a function. What is the most general form of the function $V(\phi)$ compatible with the dilatation invariance? Compute the dilatation current for such $V(\phi)$.

4. (a) Consider a field theory with three real scalar fields $\phi^a(x), a = 1, 2, 3$, and a Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi^a(x) \partial^\mu \phi^a(x) - V(\phi^a \phi^a).$$

Here summation over repeating indices a is assumed, and V is an arbitrary function. This Lagrangian is obviously invariant with respect to orthogonal transformations of the fields ϕ^a:

$$\phi^a(x) \mapsto \tilde{\phi}^a(x) = R^a_b \phi^b(x),$$

where R^a_b is a constant orthogonal 3×3 matrix. The rotation group in three dimensional space has dimension three, so we expect to get three conserved currents. Show that infintesimal transformations for $\phi^a(x)$ can be put into the form

$$\delta\phi^a(x) = \epsilon^{abc}\phi^b(x)\beta^c,$$

where $\beta^c, c = 1, 2, 3$ parametrize an infintesimal rotation, and ϵ^{abc} is a completely anti-symmetric tensor uniquely defined by the condition $\epsilon^{123} = 1$. Deduce the conserved currents corresponding to this symmetry.

(b) Let the currents found in part (a) be called $J^{a\mu}, a = 1, 2, 3$. The corresponding charges are

$$Q^a = \int d^3 x J^{a0}(x).$$

Compute the commutator of Q^a and Q^b using canonical commutation relations for ϕ^a and their time derivatives. Show that Q^a form a Lie algebra isomorphic to the Lie algebra of the rotation group (i.e. show that they obey the same commutation relations as components of the angular momentum operator in quantum mechanics).