Week 4 (due Oct. 26)

1. Consider the Maxwell theory with the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

As mentioned in class, the canonical energy-momentum tensor is not gauge-invariant. This happens because the variation of A_μ under translations,

$$\delta A_\mu = -\epsilon^\nu \partial_\nu A_\mu$$

is not gauge-invariant. One can fix this by redefining variation of A_μ as follows:

$$\tilde{\delta} A_\mu = \epsilon^\nu F_{\mu\nu} = \epsilon^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right).$$

(a) Why is it OK to change the transformation law for A_μ under translations in this way? (Hint: think gauge invariance).

(b) Derive the energy-momentum tensor $T_{\mu\nu}$ using the redefined transformation law for A_μ. (Hint: the derivation of the usual formula for the energy-momentum tensor from Noether's theorem assumes that under translations $\Delta\phi = 0$ for all fields ϕ. The modified δA_μ does not correspond to $\Delta A_\mu = 0$, so the usual expression for $T_{\mu\nu}$ does not apply.) Show that its component T_{00} is equal to $\frac{1}{2}(E^2 + B^2)$.

(c) The Maxwell Lagrangian is invariant under Lorenz transformation; the corresponding conserved current is the tensor $M^\mu_{\nu\rho}$ which is anti-symmetric in two lower indices and satisfies

$$\partial_\mu M^\mu_{\nu\rho} = 0.$$

Its component M^0_{ij} has the meaning of the density of angular momentum in the direction orthogonal to the plane ij. Derive the expression for M^0_{ij} in the Maxwell theory and show that its integral over space is gauge-invariant. (You may want to redefine variations of A_μ in a way similar to that in parts (a,b), so that $M^\mu_{\nu\rho}$ is gauge-invariant even before integration over space).

2. A one-photon state with momentum k has the form

$$\sum_{\alpha=1}^{2} f_\alpha a^\dagger_\alpha(k)|0\rangle$$

where the complex 2-component vector f_α determines the polarization state of the photon. The projection of the angular momentum of a particle on the direction of momentum is called helicity. In this problem we discuss photon states with definite helicity.

(a) Show that while the total angular momentum of a particle with spin depends on the choice of the origin of the coordinate system, the helicity is independent of this choice.

(b) Using the formula for the angular momentum found in problem 1(c), find the one-photon states which are eigenstates of the helicity operator and the corresponding helicity eigenvalues. (Hint: you are supposed to find that helicity eigenvalues are ± 1, because photons have spin 1).

3. One may consider second-quantized nonrelaivistic quantum mechanics as a quantum field theory. If the particles are noninteracting and the external potential vanishes, its action is given by

$$S = \int dt\, d^3x \left(i\Psi^\dagger \partial_t \Psi - \frac{1}{2m} \partial_j \Psi^\dagger \partial_j \Psi \right).$$

Here one can regard Ψ either as a classical field, or as a field operator.

(a) The above action has several obvious symmetries: translation in space and time, rotations, and phase symmetry:

$$\Psi \mapsto e^{i\alpha}\Psi.$$

Using Noether theorem, find the corresponding conserved currents and charges (charge is the integral of the time component of the current over space), and verify that the conserved charges agree with the usual expressions for momentum, energy, angular momentum, and particle number.

(b) Less obviously, the action is also invariant under "nonrelativistic boost" transformations

$$x^i \mapsto x^i - v^i t, \quad t \mapsto t,$$

provided the field Ψ is also transformed appropriately. Here v^i is the constant velicity vector parametrizing the boost. Find the transformation law for Ψ that achieves this, and using Noether theorem deduce the corresponding conserved current. (Hint: the answer to the first part of the question is contained, for example, in Landau and Lifshits, "Quantum mechanics", chap. III).