Week 5 (due Nov. 2)

1. Let $\xi^{\alpha\beta}$ be a Weyl bi-spinor. Let $\epsilon_{\alpha\beta} = i(\sigma_2)_{\alpha\beta}$ be an antisymmetric 2×2 matrix. Show that $\epsilon_{\alpha\beta}\xi^{\alpha\beta}$ is a scalar (i.e. it is Lorenz-invariant).

2. Consider 3-dimensional Minkowski space (two spatial dimensions, one time dimension).

(a) Construct an irreducible representation of the Dirac algebra and the corresponding spinor representation of the Lorenz group. Is the spinor representation of the Lorenz group irreducible? Show that with a suitable choice of Dirac gamma-matrices, the generators of Lorenz transformations $J_{\mu\nu}$ are purely imaginary matrices.

(b) The results of part (a) imply that matrices of Lorenz transformations $R(\Lambda)$ in the spinor representation are real matrices, so it is consistent to restrict the spinor to be real. Show that the commutation relations of Lorenz generators in 3d are the same as commutation relations of the infinitesimal generators of the group $SL(2,\mathbb{R})$ (this is the group of 2×2 real matrices with unit determinant). Show that under this identification the spinor representation of the Lorenz group corresponds to the "tautological" action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$.

(c) As explained in class, there is a 2-1 homomorphism from $SL(2,\mathbb{C})$ to the 4d Lorenz group, and this "explains" the existence of spinors. Similarly, construct a 2-1 homomorphism from $SL(2,\mathbb{R})$ to the 3d Lorenz group.