Week 7 (due Nov. 16)

1. This problem illustrates that the way parity acts on fields can not be determined on general grounds and depends on the Lagrangian of the theory.

(a) Consider the following Lagrangian for the Dirac fermion ψ:

$$L = \bar{\psi} i \partial\!\!\!/ \psi - a \bar{\psi}\psi - b \bar{\psi}\gamma_5 \psi,$$

where a and b are constants. For which values of a and b is the corresponding action Hermitian? Write down equations of motion for ψ. Show that they imply that each component of ψ satisfies the Klein-Gordon equation for some mass m. Express m in terms of a, b (assuming that a, b are chosen so that the action is Hermitian).

(b) The results of part (a) suggest that the above theory is equivalent to the ordinary Dirac theory with the Lagrangian

$$L_0 = \bar{\psi} i \partial\!\!\!/ \psi - m \bar{\psi}\psi$$

Show that this is the case by finding a linear change of fields variables $\psi = S\psi'$, where the matrix S chosen so that the action for ψ' is given by L_0. (Hint: it may be easier to find the desired change of variables by working with a pair of Weyl spinors which make up the Dirac spinor).

(c) Show that the action L is not invariant under the usual parity transformation for ψ discussed in class. On the other hand, the results of part (b) imply that the usual parity transformation of ψ' is a symmetry of the theory. Find how the latter transformation acts on ψ.

2. (a) Consider a theory with N Dirac fields $\psi^i, i = 1, \ldots, N$ and a Lagrangian

$$L = \bar{\psi}_i i \partial\!\!\!/ \psi^i - m \bar{\psi}_i \psi^i.$$

Show that the action has a continuous $U(N)$ symmetry given by $\psi^i \mapsto U^i_j \psi^j$, where U is a unitary matrix. Find the corresponding conserved currents.

(b) Now let us set $m = 0$. Show that the symmetry is now enhanced to $U(N) \times U(N)$. (Hint: it is easier to see this if one writes the action in terms of two-component Weyl spinors. Then the two $U(N)$'s appear because left-handed and right-handed Weyl spinors can be transformed independently. Such symmetries are called chiral symmetries.) Find the corresponding currents (in the four-component notation). Compute their divergence both when $m = 0$ and $m \neq 0$ and verify that it is zero in the former case and nonzero in the latter case.