Week 8 (due May 24)

1. Let G be a gauge group (you can think of G as a subgroup of $U(N)$ for some N). Let W be the Lie algebra of G (which is some linear subspace of the space of Hermitian $N \times N$ matrices). Let w be the dimension of W. Let t^a be some basis of W normalized so that $\operatorname{Tr} t^a t^b = \frac{1}{2}\delta^{ab}$. The structure constants of W are defined by the equation

$$[t^a, t^b] = i f^{abc} t^c.$$

Consider a unitary representation R of the group G and the corresponding representation of its Lie algebra W. This means that for any $g \in G$ we are given an $M \times M$ unitary matrix $R(g)$ so that $R(g \cdot g') = R(g) \cdot R(g')$. Also, if we consider g which are infinitesimally close to 1, i.e. $g \simeq 1 + i\alpha^a t^a$ for some real numbers α^a, then $R(g) \simeq 1 + i\alpha^a T^a$, for some $M \times M$ Hermitian matrices T^a.

(a) Show that T^a obey the same commutation relations as t^a, i.e.

$$[T^a, T^b] = i f^{abc} T^c.$$

(Hint: consider the commutator $g_1 g_2 g_1^{-1} g_2^{-1}$ of group elements $g_1, g_2 \in G$ where both g_1 and g_2 are close to 1).

(b) Show that the matrix $T^a T^a$ (summation over a is assumed) commutes with T^b for any b. A representation of the Lie algebra W is called irreducible if any matrix which commutes with all T^b is proportional to the identity matrix. Let us assume that the representation we are dealing is irreducible. Then $T^a T^a = C_2(R) \cdot 1$, where $C_2(R)$ is a real number. $C_2(R)$ is called the second Casimir of the representation R. Compute the second Casimir for the "defining" representation of W given by the matrices t^a.

(c) Let the field ψ take values in the representation R. That is, it is an M-component field with components $\psi^i(x), i = 1, \ldots, M$ which under a gauge transformation $g(x)$ transforms into $\tilde{\psi}(x)$ with components $\tilde{\psi}^i = R(g)^i_j \psi^j(x)$. The covariant derivative of ψ is defined by $D_\mu \psi = (\partial_\mu - iT^a A^a_\mu)\psi$. Show that under an infinitesimal gauge transformation $g(x) \simeq 1 + i\alpha^a(x)t^a$ the covariant derivative changes into $D_\mu\psi + i\alpha^a(x)T^a D_\mu\psi$. (You will need the transformation law for $A_\mu = A^a_\mu t^a$ which reads $\delta A_\mu = \partial_\mu \alpha + i[\alpha, A_\mu]$ where $\alpha = \alpha^a t^a$).

(d) Instead of working with $N \times N$ gauge field $A_\mu = A^a_\mu t^a$, one can work with $M \times M$ gauge field $B_\mu = A^a_\mu T^a$. The information contained in B_μ is, of course, the same as in A_μ. In terms of B_μ the covariant derivative of ψ is simply $D_\mu\psi = \partial_\mu\psi - iB_\mu\psi$. Let us call this covariant derivative D^B. Let us define the new field strength $F^B_{\mu\nu}$ by

$$[D^B_\mu, D^B_\nu] = -iF^B_{\mu\nu}$$

Show that $F^B_{\mu\nu} = T^a F^a_{\mu\nu}$. Further, let us define the Yang-Mills action using F^B instead of F, i.e. let

$$L_{YM} = -\frac{1}{2g_B^2}\operatorname{Tr} F^B_{\mu\nu}F^{B\mu\nu}.$$

This action is, of course, identical to the one written in terms of $F_{\mu\nu}$, provided g_B is related to g in some way. What is the relation between g_B and g? (Express the relation in terms of the second Casimir $C_2(R)$ of the representation R, its dimension M and the dimension w of W).

(e) The Maxwell equations of motion read $\partial_\mu F^{\mu\nu} = j^\nu$, where j^ν is the current density. Contracting them with ∂_ν and using anti-symmetry of $F_{\mu\nu}$, we learn that j^ν must be conserved, i.e. $\partial_\nu j^\nu = 0$. Consider now the Yang-Mills equations of motion $D_\mu F^{\mu\nu} = J^\nu$, where J^ν is again the current density. Unlike the abelian case, J^ν is valued in the Lie algebra W (i.e. it is an $N \times N$ matrix). How should J^ν transform under infinitesimal gauge transformations? How should one define its covariant derivative? Show that the Yang-Mills equations of motion imply $D_\nu J^\nu = 0$.