Winter quarter - Week 2 (due Jan. 18)

1. Consider the following Lagrangian for a real scalar field in 4d space-time:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3.$$

(a) Write down Feynman rules for this theory (both in coordinate space and momentum space).

(b) Consider 2-2 scattering in this theory (i.e. two incoming and two outgoing particles). Let the initial particles have 4-momenta p_1, p_2 and the final ones k_1, k_2. Draw all Feynman diagrams which contribute to this scattering process at leading order in λ. Show that the corresponding matrix element is of order λ^2.

(c) Write down the analytic expressions for all the diagrams in part (b), sum them up to get the complete matrix element at this order in λ, and compute the cross-section for the 2-2 scattering. Express it in terms of the scattering angle (i.e. the angle between the spatial parts of p_1 and k_1) and the center-of-mass energy of the colliding particles.

2. Consider a Feynman diagram contributing to the S-matrix in the ϕ^4 theory. Let the diagram have V interaction vertices, I internal edges (propagators) and E external edges (corresponding to incoming and outgoing particles).

(a) Show that the following identity holds: $4V = E + 2I$.

(b) Let L be the number of loops in the diagram. That is, the net number of independent 4-momenta on internal lines which have to be integrated over after all conservation laws have been taken into account. Show that $L = I - V + 1$. (Remark for the mathematically-minded: from the mathematical viewpoint, L is the first Betti number of the Feynman diagram, and the above identity is a form of Euler theorem for graphs). Use this identity and the result of part (a) to express L in terms of E and V alone.

(c) Determine the mass dimension of the analytic expression for the Feynman diagram. Upon using the results of parts (a) and (b), show that it is given by $D = 4 - E$. (Remarkably, D does not depend on the number of vertices, but only on the number of external edges. This will become important when we learn how to deal with the divergences arising from integration over internal 4-momenta.)