# Week 2 (due Jan. 23)

Reading: Srednicki, sections 33,34,35.

1. The Hodge star operator on antisymmetric second rank tensors is defined as

$$(\star a)_{\mu\nu} = \frac{1}{2} g_{\mu\rho} g_{\nu\sigma} \epsilon^{\rho\sigma\alpha\beta} a_{\alpha\beta}.$$

Here $g_{\mu\nu}$ is either Minkowski metric or Euclidean metric, and $\epsilon^{\mu\nu\rho\sigma}$ is a completely antisymmetric tensor such that $\epsilon^{0123} = 1$. Show that the Hodge star satisfies $\star\star = -1$ in Minkowski space and $\star\star = 1$ in Euclidean space.

2. (ab) Consider Lorenz group in three-dimensional space-time (i.e. one timelike direction, two spacelike directions). Show that the group is three-dimensional. Construct a 2-1 homomorphism from $SL(2, \mathbb{R})$ (the group of real $2\times 2$ matrices with unit determinant) to the 3d Lorenz group. This shows that representations of $SL(2, \mathbb{R})$ can be thought of as projective representations of the 3d Lorenz group. The tautological 2-dimensional representation of $SL(2, \mathbb{R})$ can be taken as the spinor representation. It is obviously real. Is it self-dual? How many inequivalent spinor representations are there in 3d?

3. Lorenz transformations and translations together generate the group called the Poincare group. The nonrelativistic analog of the Poincare group is the Galileo group. For simplicity, let us consider the case when there is only one spacelike dimension with coordinate $x$, as well as time $t$. The Galileo group is generated by spatial and time translations, as well as nonrelativistic boosts:

$$x \to x - Vt, \quad t \to t.$$

Here $V$ parameterizes the boost. Let $P, H$, and $K$ be generators of infinitesimal space translations, time translations, and boosts. For example, $P = -i\partial_x$ and $H = i\partial_t$.

(a) What does $K$ look like? What are the commutation relations for $P, H$ and $K$?

(b) Consider now the Schrodinger equation for a free nonrelativistic particle on a line:

$$i\partial_t \Psi(x,t) = -\frac{1}{2m} \partial_x^2 \Psi(x,t).$$

This equation is not invariant under the naive substitution $\Psi(x,t) \mapsto \Psi(x - Vt, t)$. Nevertheless, Galilean invariance can be restored if we modify the transformation law for $\Psi$ (see problem 4.3 from the fall quarter):

$$\Psi(x,t) \mapsto \Psi(x - Vt, t) \exp\left(imVx - imV^2 t/2\right).$$

Determine the generator of infinitesimal boosts acting on $\Psi$. Compute the algebra of $K, P, H$. Show that the commutator of $K$ and $P$ differs from the one in part (a) by a "central" term (i.e. by a term proportional to the identity operator).

(c) The most general Galilean transformation looks as follows:

$$t \to t + a, \quad x \to x - Vt + b.$$

Let us define the corresponding transformation of the wavefunction as the composition of (1) boost as found in part (b); (2) the usual time and space translation $\Psi(x,t) \mapsto \Psi(x+b, t+a)$. Now consider doing spatial translation and then boost. From the viewpoint of the Galileo group, this is the same as first doing boost and then spatial translation. Show that nevertheless the transformation of the wavefunction depends on the order of these operations. Deduce from this that the Hilbert space of the free particle is a *projective* representation of the Galileo group. Determine the corresponding "cocycle", i.e. a function $\phi(g_1, g_2)$ such that transformation operators $R(g)$ on the Hilbert space satisfy

$$R(g_1)R(g_2) = e^{i\phi(g_1,g_2)}R(g_1 \cdot g_2)$$

for any two elements $g_1, g_2$ of the Galileo group $G$. For simplicity, you may restrict to the subgroup of the Galileo group generated by spatial translations and boosts.