Week 6 (due Feb. 15)

1. If you get stuck, consult Feynman and Hibbs, "Quantum Mechanics And Path Integrals", chapter 3, or Shankar, "Principles of Quantum Mechanics".

(a) Consider the following Lagrangian:

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 + fq(t),$$

where f is a constant. It describes a particle moving under the influence of a constant force f. The path-integral computing the transition amplitude from $q(0) = q_a$ to $q(T) = q_b$ is

$$K(q_a, q_b, T) = \lim_{N\to\infty} C(N) \int \prod_{i=1}^{N-1} dq_i \exp\left[i \sum_{i=1}^{N} \epsilon \left(\frac{m}{2} \frac{(q_i - q_{i-1})^2}{\epsilon^2} + f q_{i-1} \right) \right],$$

where $\epsilon = T/N$ and

$$C(N) = \left(\frac{2\pi i \epsilon}{m}\right)^{-N/2}.$$

Note that the integral over each q_i is Gaussian, so one can do the integral over q_1, say. Show that result is a Gaussian integral w. r. to q_2, so one can integrate again, etc.. Show that the resulting transition amplitude is

$$K(q_a, q_b, T) = \left(\frac{m}{2\pi i T}\right)^{1/2} \exp i \left[\frac{m(q_b - q_a)^2}{2T} + \frac{fT}{2}(q_a + q_b) - \frac{f T^2}{24} \right]. \quad (1)$$

While computing Gaussian integrals, make sure that you get overall numerical factors right (even though we were sloppy about them in class).

(b) Given q_b and q_a, one can also consider solutions of classical equations of motion such that $q(0) = q_a$, $q(T) = q_b$. Such a solution is called a classical trajectory from q_a to q_b. Show that in the present case there is a unique classical trajectory from q_a to q_b and determine its explicit form. Let S_{cl} be the value of the classical action $S[q(t)]$ on the classical trajectory $q_{cl}(t)$. Show that the expression in square brackets in Eq. (1) is precisely S_{cl}. Can you explain this "coincidence"?

2. (a) Consider a quadratic functional

$$F[\phi(x)] = \int \partial_\mu \phi(x') \partial_\nu \phi(y') K^{\mu\nu}(x', y') d^4x' d^4y',$$

where $K^{\mu\nu}(x,y)$ is some fixed matrix function on $\mathbb{R}^4 \times \mathbb{R}^4$. Compute the second functional derivative

$$\frac{\delta^2 F[\phi]}{\delta\phi(x)\delta\phi(y)}.$$

(b) Let $u(t)$ be a function of one variable. Consider the functional

$$F[u(t)] = u(0)u''(0).$$

Evaluate the functional derivatives

$$\frac{\delta F}{\delta u(t)}, \quad \frac{\delta^2 F}{\delta u(t)\delta u(t')}.$$

3. Consider the Lagrangian for the massive vector field A_μ:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu.$$

The second term in the action destroys gauge-invariance, so there is no need for gauge-fixing. The propagator can be found by inverting an appropriate differential operator acting on A_μ.

Find the propagator for the field A_μ in momentum space. What happens with it when one takes the limit $m \to 0$? Should one take m^2 to be positive or negative?