Week 7 (due Feb. 22)

1. Consider the following Lagrangian for bosonic degrees of freedom $q_1(t), q_2(t)$ and fermionic degrees of freedom $\bar{\theta}(t), \theta(t)$:

$$L = \frac{1}{2}\sum_{i=1}^{2} \dot{q}_i{}^2 + i\bar{\theta}\dot{\theta} + \sum_i A_i(q)\dot{q}_i(t) - f(q)\bar{\theta}\theta.$$

Here $A_i(q)$ is an arbitrary vector function of q_1, q_2, and $f(q)$ is an arbitrary scalar function.

(a) Write down canonical (anti)commutation relations for this system and compute the Hamiltonian.

(b) Show that upon quantization the Hamiltonian becomes equivalent to the Hamiltonian of a spin-1/2 charged particle moving in a plane under the influence of a vector potential $A_i(q)$, while $f(q)$ plays the role of the component of the magnetic field in the direction perpendicular to the plane. (Note though that $f(q)$ is not related in any way to $A_i(q)$, while for a charged particle the magnetic field $B_3 = \partial_1 A_2 - \partial_2 A_1$.)

2. Suppose we are given a quantum-mechanical system whose Hilbert space V is a sum of two Hilbert spaces: $V = V_+ \oplus V_-$. An operator acting on V is called even if it maps V_+ to V_+ and V_- to V_-. An operator is called odd if it maps V_+ to V_- and V_- to V_+. Such a system is called supersymmetric if its Hamiltonian H is even and there exists an odd operator Q such that

$$Q^2 = 0, \quad QQ^\dagger + Q^\dagger Q = 2H.$$

Q is called the supercharge.

Consider the Pauli Hamiltonian for an electron moving on a plane under the influence of a vector potential A_i, $i = 1, 2$. Keep the gyromagnetic ratio arbitrary for the moment. We let V_+ (resp. V_-) be the subspace of the Hilbert space consisting of states with spin up (resp. down). That is, V_+ is the eigenspace of σ_3 with eigenvalue $+1$, while V_- is the eigenspace of σ_3 with eigenvalue -1. Consider the following odd operator:

$$Q = \frac{1}{2}(\sigma_1 + i\sigma_2)\left(p_1 - A_1(q) - i(p_2 - A_2(q)\right),$$

where $p_k = -i\partial/\partial q_k$, as usual. Verify that the Pauli Hamiltonian is supersymmetric for one particular value of the gyromagnetic ratio, if one takes Q as the supercharge. What is this special value of the gyromagnetic ratio? (Note that we set the mass of the electron to be 1).