9.5 Path Integrals for Fermions

We now turn to the problem of extending the path-integral formalism to cover theories containing fermions as well as bosons. It would be easy to proceed in a purely formal way, by analogy with the bosonic case, with the justification that this gives the 'right' Feynman rules. Instead, we will here derive the path-integral formalism for fermions directly from the principles of quantum mechanics, as we did for bosons.[9]

As before, we will start with a general quantum mechanical system, with 'coordinates' Q_a and canonical conjugate 'momenta' P_a, but now satisfying anticommutation rather than commutation relations:

$$\left\{Q_a, P_b\right\} = i\,\delta_{ab}\,, \tag{9.5.1}$$

$$\left\{Q_a, Q_b\right\} = \left\{P_a, P_b\right\} = 0\,. \tag{9.5.2}$$

(These are Schrödinger-picture operators, or in other words Heisenberg-picture operators at time $t = 0$.) Later we will replace the discrete index a with a spatial position $\mathbf{x}$ and a field index m.

We wish first to construct a complete basis for the states on which the Qs and Ps act. Note that for any given a, we have

$$Q_a^2 = P_a^2 = 0\,. \tag{9.5.3}$$

It follows that there will always be a 'ket' state $|0\rangle$ annihilated by all Q_a:

$$Q_a|0\rangle = 0\,, \tag{9.5.4}$$

and a 'bra' state $\langle 0|$ annihilated (from the right) by all P_a:

$$\langle 0|\,P_a = 0\,. \tag{9.5.5}$$

For instance, we can take

$$|0\rangle \propto \left(\prod_a Q_a\right)|f\rangle\,, \qquad \langle 0| \propto \langle g|\left(\prod_a P_a\right)\,,$$

where $|f\rangle$ and $\langle g|$ are any kets and bras for which these expressions do not vanish. (They cannot vanish for all $|f\rangle$ and $\langle g|$ unless the operators $\prod_a Q_a$ and $\prod_a P_a$ vanish, which we assume not to be the case.) These states satisfy Eqs. (9.5.4) and (9.5.5) by virtue of Eq. (9.5.3). They are not in general unique, because there may be other bosonic degrees of freedom that distinguish the various possible $|0\rangle$ and $\langle 0|$, but for simplicity we will limit ourselves here to the case where the only degrees of freedom are those described by the fermionic operators Q_a and P_a, and will assume that the states satisfying Eqs. (9.5.4) and (9.5.5) are unique up to constant factors, which we choose so that

$$\langle 0|0\rangle = 1\,. \tag{9.5.6}$$

(Note that this normalization convention could not be imposed if we had defined $\langle 0|$ as the left-eigenstate of the Q_a with eigenvalue zero, because in this case $\langle 0|\{Q_a, P_b\}|0\rangle$ would vanish, which with Eq. (9.5.1) would imply that $\langle 0|0\rangle = 0$.)

As we saw in Section 7.5, in the Dirac theory Q_a is not Hermitian, but instead has an adjoint $-iP_a$, in which case $\langle 0|$ *can* be regarded as simply the adjoint of $|0\rangle$. However, there are fermionic operators (such as the 'ghost' fields to be introduced in Volume II) for which P_a is unrelated to the adjoint of Q_a. In what follows we will not need to assume anything about the adjoints of Q_a or P_a, or about any relation between $|0\rangle$ and $\langle 0|$.

A complete basis for the states of this system is provided by $|0\rangle$ and the states (antisymmetric in indices $a, b, \cdots$)

$$|\,a, b, \cdots\rangle \equiv P_a\, P_b \cdots |0\rangle \tag{9.5.7}$$

with any number of *different* Ps acting on $|0\rangle$. That is, the result of acting on these states with any operator function of the Ps and Qs can be written as a linear combination of the same set of states. In particular, if an index a is unequal to any of the indices appearing in $|\,b, c, \cdots\rangle$, then

$$Q_a|\,b, c, \cdots\rangle = 0\,, \tag{9.5.8}$$

$$P_a|\,b, c, \cdots\rangle = |\,a, b, c \cdots\rangle\,. \tag{9.5.9}$$

On the other hand, if a *is* equal to one of the indices in the sequence, $b, c, \cdots$, we can always rewrite the state (possibly changing its sign) so that a is the first of these indices, in which case we have

$$Q_a|\,a, b, c, \cdots\rangle = i\,|\,b, c \cdots\rangle\,, \tag{9.5.10}$$

$$P_a|\,a, b, c, \cdots\rangle = 0\,. \tag{9.5.11}$$

Similarly, we may define a complete dual basis, consisting of $\langle 0|$ and the states (also antisymmetric in the indices)

$$\langle a, b, \cdots|\equiv \langle 0|\cdots(-iQ_b)(-iQ_a)\,. \tag{9.5.12}$$

Using Eqs. (9.5.4)–(9.5.6) and the anticommutation relation (9.5.1), we see that the scalar products of these states take the values

$$\langle c, d, \cdots|\,a, b, \cdots\rangle = \langle 0|\cdots(-iQ_d)(-iQ_c)\,P_aP_b\cdots|0\rangle$$
$$= \begin{cases} 0 & \text{if } \{c, d, \cdots\} \neq \{a, b, \cdots\} \\ 1 & \text{if } c = a,\ d = b, \text{ etc.} \end{cases} \tag{9.5.13}$$

where $\{\cdots\}$ here denotes the set of indices within the brackets, irrespective of order.

In deriving the Feynman rules, we would like to be able to rewrite sums over intermediate states like (9.5.7) as integrals over eigenstates of the Q_a or the P_a. However, it is not possible for these operators to have eigenvalues (other than zero) in the usual sense. Suppose we try to find a state $|q\rangle$ that satisfies (for all a)

$$Q_a|q\rangle = q_a|q\rangle \,. \tag{9.5.14}$$

From Eq. (9.5.2) we see that

$$q_a q_b + q_b q_a = 0 \tag{9.5.15}$$

which is impossible for ordinary numbers. However, nothing can stop us from introducing an algebra of 'variables' (known as *Grassmann variables*) q_a, which act like c-numbers as far as the physical Hilbert space is concerned, but which still satisfy the anticommutation relations (9.5.15). We will require further that

$$\{q_a, q'_b\} = \{q_a, Q_b\} = \{q_a, P_b\} = 0, \tag{9.5.16}$$

where q and q' denote any two 'values' of these variables. We can now construct eigenstates $|q\rangle$ satisfying Eq. (9.5.14):

$$|q\rangle = \exp\left(-i\sum_a P_a q_a\right)|0\rangle \tag{9.5.17}$$

with the exponential defined as usual by its power series expansion. (To verify Eq. (9.5.14), use the fact that all $P_a q_a$ commute with one another and have zero square, so that

$$\begin{aligned}
[Q_a - q_a]\,|q\rangle &= [Q_a - q_a]\,\exp(-iP_a q_a)\,\exp\left(-i\sum_{b\neq a} P_b q_b\right)|0\rangle \\
&= [Q_a - q_a]\,[1 - iP_a q_a]\,\exp\left(-i\sum_{b\neq a} P_b q_b\right)|0\rangle \\
&= [-i\{Q_a, P_a\}q_a - q_a]\,\exp\left(-i\sum_{b\neq a} P_b q_b\right)|0\rangle = 0
\end{aligned}$$

as required by Eq. (9.5.14).) We can also define left-eigenstates $\langle q|$ (*not* the adjoints of $|q\rangle$), as

$$\langle q| \equiv \langle 0|\left(\prod_a Q_a\right)\exp\left(-i\sum_a q_a P_a\right) = \langle 0|\left(\prod_a Q_a\right)\exp\left(+i\sum_a P_a q_a\right), \tag{9.5.18}$$

where $\prod_a$ is the product in whatever order we take as standard. By the same argument as for Eq. (9.5.14), we see that

$$\langle q|\,Q_a = \langle q|q_a \,. \tag{9.5.19}$$

These eigenstates have the scalar product

$$\langle q'|q\rangle = \langle 0|\left(\prod_a Q_a\right)\exp\left(i\sum_b P_b(q'_b - q_b)\right)|0\rangle$$
$$= \langle 0|\left(\prod_a Q_a\right)\left(\prod_b (1 + iP_b(q'_b - q_b))\right)|0\rangle \,.$$

Moving each Q_a to the right (starting with the rightmost) yields factors $i^2(q'_a - q_a)$, which we move to the right out of the scalar product, so

$$\langle q'|q\rangle = \prod_a (q_a - q'_a)\,. \tag{9.5.20}$$

We shall see that Eq. (9.5.20) plays the role of a delta function in integrals over the qs.

In the same way, we can construct right- and left-eigenstates of the P_a:

$$P_a|p\rangle = p_a|p\rangle\,, \tag{9.5.21}$$
$$\langle p|\,P_a = \langle p|p_a\,, \tag{9.5.22}$$

where the p_a are like q_a anticommuting c-numbers (taken for convenience to anticommute with the q_a and all fermionic operators as well as each other), and

$$|p\rangle = \exp\left(-i\sum_a Q_a p_a\right)\left(\prod_b P_b\right)|0\rangle\,, \tag{9.5.23}$$

$$\langle p| = \langle 0|\,\exp\left(-i\sum_a p_a Q_a\right) \tag{9.5.24}$$

with scalar product (now derived by moving the Ps to the left)

$$\langle p'|p\rangle = \prod_a (p'_a - p_a)\,. \tag{9.5.25}$$

The scalar products of these two sorts of eigenstate with each other are

$$\langle q|p\rangle = \langle q|\exp\left(-i\sum_a Q_a p_a\right)\left(\prod_a P_a\right)|0\rangle$$
$$= \left(\prod_a \exp(-iq_a p_a)\right)\langle q|\left(\prod_a P_a\right)|0\rangle$$
$$= \left(\prod_a \exp(-iq_a p_a)\right)\langle 0|\left(\prod_a Q_a\right)\left(\prod_a P_a\right)|0\rangle$$

and so

$$\langle q|p\rangle = \chi_N \exp\Big(-i\sum_a q_a p_a\Big) = \chi_N \exp\Big(i\sum_a p_a q_a\Big)\,, \tag{9.5.26}$$

where χ_N is a phase that depends only on the number N of Q_a operators:

$$\chi_N \equiv \langle 0| \left(\prod_a Q_a\right) \left(\prod_a P_a\right) |0\rangle = i^N(-1)^{N(N-1)/2} .$$

Somewhat more simply, we also find

$$\langle p|q\rangle = \prod_a \exp(-ip_a q_a) . \tag{9.5.27}$$

It is easy to see that the states $|q\rangle$ are in a sense a complete set (and so also are the $|p\rangle$.) From the definitions (9.5.17), we see that the state $|a, b, \cdots\rangle$ in the general basis is (up to a phase) just the coefficient of the product $q_a q_b \cdots$ in an expansion of $|q\rangle$ in a sum of products of qs. Therefore we can write any state $|f\rangle$ in the form

$$|f\rangle = f_0|q\rangle_0 + \sum_a f_a|q\rangle_a + \sum_{a\neq b} f_{ab}|q\rangle_{ab} + \cdots ,$$

where the fs are numerical coefficients, and a subscript $a, b, \cdots$ on $|q\rangle$ denotes the coefficient of $q_a q_b \cdots$ in $|q\rangle$.

In summing over states, it will be very convenient to introduce a sort of integration over fermionic variables, known as *Berezin integration*,[10] that is designed to pick out the coefficients of such products of anticommuting c-numbers. For any set of such variables ξ_n (either ps or qs or both together), the most general function $f(\xi)$ (either a c-number or a state-vector like $|q\rangle$) can be put in the form

$$f(\xi) = \left(\prod_n \xi_n\right) c \;+\; \text{terms with fewer } \xi \text{ factors} \tag{9.5.28}$$

and the integral over the ξs is defined simply by

$$\int \left(\tilde{\prod_n} d\xi_n\right) f(\xi) \equiv c \tag{9.5.29}$$

with the tilde in Eq. (9.5.29) indicating that we use the convenient convention that the differentials are written in an order *opposite* to that of the product of integration variables in Eq. (9.5.28). Since this product is antisymmetric under the interchange of any two ξs, the integral is likewise antisymmetric under the interchange of any two $d\xi$s, so these 'differentials' effectively anticommute

$$d\xi_n d\xi_m + d\xi_m d\xi_n = 0 . \tag{9.5.30}$$

Also, the coefficient c may itself depend on other unintegrated c-number variables that anticommute with the ξs over which we integrate, in which case it is important to standardize the definition of c by moving all ξs to the left of c before integrating over them, as we have done in Eq. (9.5.28).

For instance, the most general function of a pair of anticommuting c-numbers ξ_1 and ξ_2 takes the form

$$f(\xi_1,\xi_2) = \xi_1\xi_2 c_{12} + \xi_1 c_1 + \xi_2 c_2 + d$$

because the squares and all higher powers of ξ_1 and ξ_2 vanish. This function has the integrals

$$\int d\xi_1\, f(\xi_1,\xi_2) = \xi_2 c_{12} + c_1\,, \quad \int d\xi_2\, f(\xi_1,\xi_2) = -\xi_1 c_{12} + c_2\,,$$

$$\int d\xi_2\, d\xi_1\, f(\xi_1,\xi_2) = c_{12}\,.$$

Note that the multiple integral is the same as a repeated integral:

$$\int d\xi_2\, d\xi_1\, f(\xi_1,\xi_2) = \int d\xi_2 \Big[\int d\xi_1\, f(\xi_1,\xi_2) \Big]\,,$$

a result that can easily be extended to integrals over any number of fermionic variables. (It was in order to obtain this result without extra sign factors that we took the product of differentials in Eq. (9.5.29) to be in the opposite order to the product of variables in Eq. (9.5.28).) Indeed, we could have first defined the integral over a single anticommuting c-number ξ_1, and then defined multiple integrals in the usual way by iteration. The most general function of anticommuting c-numbers is linear in any one of them

$$f(\xi_1,\xi_2,\cdots) = b(\xi_2\cdots) + \xi_1 c(\xi_2\cdots)$$

(because $\xi_1^2 = 0$), and its integral over ξ_1 is defined as

$$\int d\xi_1 f(\xi_1,\xi_2,\cdots) = c(\xi_2,\cdots)\,.$$

Repeating this process leads to the same multiple integral as defined by Eqs. (9.5.28) and (9.5.29).

This definition of integration shares some other properties with multiple integrals (from $-\infty$ to $+\infty$) over ordinary real variables, but there are significant differences.

Obviously, Berezin integration is linear, in the sense that

$$\int \left(\widetilde{\prod_n} d\xi_n\right) \Big[f(\xi)+g(\xi)\Big] = \int \left(\widetilde{\prod_n} d\xi_n\right) f(\xi) + \int \left(\widetilde{\prod_n} d\xi_n\right) g(\xi) \quad (9.5.31)$$

and also

$$\int \left(\widetilde{\prod_n} d\xi_n\right) \Big[f(\xi)a(\xi')\Big] = \left[\int \left(\widetilde{\prod_n} d\xi_n\right) f(\xi)\right] a(\xi')\,, \quad (9.5.32)$$

where $a(\xi')$ is any function (including a constant) of any anticommuting c-numbers ξ'_m over which we are *not* integrating. However, linearity with respect to left-multiplication is not so obvious. If we are integrating over ν variables, then since ξ'_m is assumed to anticommute with all ξ_n, we have

$$a\Big((-)^\nu \xi'\Big)\left(\prod_n \xi_n\right) = \left(\prod_n \xi_n\right) a(\xi')$$

and so

$$\int \left(\prod_n d\xi_n\right)\Big[a\Big((-)^\nu \xi'\Big) f(\xi)\Big] = a(\xi') \int \left(\prod_n d\xi_n\right) f(\xi)\,. \qquad (9.5.33)$$

It is therefore very convenient (though not strictly necessary) to take the differentials $d\xi_n$ to anticommute with all anticommuting variables (including the ξ_n):

$$(d\xi_n)\xi'_m + \xi'_m(d\xi_n) = 0 \qquad (9.5.34)$$

in which case Eq. (9.5.33) reads more simply

$$\int a(\xi')\left(\prod_n d\xi_n\right) f(\xi) = a(\xi') \int \left(\prod_n d\xi_n\right) f(\xi)\,. \qquad (9.5.35)$$

Another similarity with ordinary integration is that, for an arbitrary anticommuting c-number ξ' independent of ξ,

$$\int \left(\prod_n d\xi_n\right) f(\xi + \xi') = \int \left(\prod_n d\xi_n\right) f(\xi) \qquad (9.5.36)$$

since shifting ξ by a constant only affects the terms in f with fewer than the total number of ξ-variables.

On the other hand, consider a change of variables

$$\xi_n \to \xi'_n = \sum_m \mathscr{S}_{nm}\xi_m\,, \qquad (9.5.37)$$

where $\mathscr{S}$ is an arbitrary non-singular matrix of ordinary numbers. The product of the new variables is

$$\prod_n \xi'_n = \sum_{m_1 m_2 \cdots} \left(\prod_n \mathscr{S}_{n m_n}\xi_{m_n}\right).$$

But $\prod_n \xi_{m_n}$ here is just the same as the product (in the original order) $\prod_n \xi_n$, except for a sign $\epsilon[m]$ which is $+1$ or -1 according to whether the permutation $n \to m_n$ is an even or odd permutation of the original order:

$$\prod_n \xi'_n = \left[\sum_{m_1 m_2 \cdots} \left(\prod_n \mathscr{S}_{n m_n}\right)\epsilon[m]\right] \prod_n \xi_n = (\mathrm{Det}\,\mathscr{S}) \prod_n \xi_n\,.$$

This applies whatever order we take for the ξ_n, as long as we take the ξ'_n in the same order. It follows that the coefficient of $\prod_n \xi'_n$ in any function

$f(\xi)$ is just $(\text{Det}\,\mathscr{S})^{-1}$ times the coefficient of $\prod_n \xi_n$, a statement we write as

$$\int \left(\widetilde{\prod_n} d\xi'_n\right) f = (\text{Det}\,\mathscr{S})^{-1} \int \left(\widetilde{\prod_n} d\xi_n\right) f\,. \tag{9.5.38}$$

This is the usual rule for changing variables of integration, except that $(\text{Det}\,\mathscr{S})$ appears to the power -1 instead of $+1$. We shall use Eq. (9.5.38) and the linearity properties (9.5.31), (9.5.32), and (9.5.35) later to evaluate the integrals encountered in deriving the Feynman rules for theories with fermions.

We can now use this definition of integration to write the completeness condition as a formula for an integral over eigenvalues. As already mentioned, any state $|f\rangle$ can be expanded in a series of the states $|0\rangle$, $|a\rangle$, $|a,b\rangle$, etc. and these states are (up to a phase) the coefficients of the products 1, q_a, $q_a q_b$, etc. in the Q-eigenstate $|q\rangle$. According to the definition of integration here, we can pick out the coefficient of any product $q_b q_c q_d \cdots$ in the state $|q\rangle$ by integrating the product of $|q\rangle$ with all q_a with *a not* equal to $b, c, d, \cdots$. Thus, by choosing a function $f(q)$ as a suitable sum of such products of qs, we can write any state $|f\rangle$ as an integral:

$$|f\rangle = \int \left(\widetilde{\prod_a} dq_a\right) |q\rangle f(q) = \int |q\rangle \left(\widetilde{\prod_a} dq_a\right) f(q)\,. \tag{9.5.39}$$

(We can move $|q\rangle$ to the left of the differentials without any sign changes because the exponential in Eq. (9.5.17) used to define $|q\rangle$ involves only even numbers of fermionic quantities.) To find the function $f(q)$ for a given state-vector $|f\rangle$, take the scalar product of Eq. (9.5.39) with some bra $\langle q'|$ (with q' any fixed Q-eigenvalue). According to Eqs. (9.5.35) and (9.5.20), this is

$$\langle q'|f\rangle = \int \left(\prod_a (q_a - q'_a)\right) \left(\widetilde{\prod_b} dq_b\right) f(q)\,.$$

Moving every factor $(q_a - q'_a)$ to the right past every differential dq_b yields a sign factor $(-)^{N^2} = (-)^N$, where N is now the total number of q_a variables, so

$$\langle q'|f\rangle = (-)^N \int \left(\widetilde{\prod_b} dq_b\right) \left(\prod_a (q_a - q'_a)\right) f(q)\,.$$

We can rewrite $f(q)$ as $f(q' + (q - q'))$ and expand in powers of $q - q'$. All terms beyond the lowest order vanish when multiplied with the product