$\prod(q_a - q'_a)$, so

$$\left(\prod_a (q_a - q'_a)\right) f(q) = \left(\prod_a (q_a - q'_a)\right) f(q') \,, \qquad (9.5.40)$$

which partly justifies our earlier remark that Eq. (9.5.20) plays the role of a delta function for integrals over the qs. Using Eq. (9.5.32), we now have

$$\langle q'|f\rangle = (-)^N \left[\int \left(\tilde{\prod_b} dq_b\right)\left(\prod_a (q_a - q'_a)\right) f(q')\right] .$$

The term in the integrand proportional to $\prod q_a$ has coefficient $f(q')$, so according to our definition of integration $\langle q'|f\rangle = (-)^N f(q')$. Inserting this back in Eq. (9.5.39) gives our completeness relation

$$|f\rangle = (-)^N \int |q\rangle \left(\tilde{\prod_b} dq_b\right) \langle q|f\rangle \,,$$

or as an operator equation

$$1 = \int |q\rangle \left(\tilde{\prod_a} -dq_a\right) \langle q| \,. \qquad (9.5.41)$$

In exactly the same way, we can also show that

$$1 = \int |p\rangle \left(\tilde{\prod_a} dp_a\right) \langle p| \,. \qquad (9.5.42)$$

We are now in a position to calculate transition matrix elements. As before, we define time-dependent operators

$$Q_a(t) \equiv \exp(iHt)\, Q_a \exp(-iHt) \qquad (9.5.43)$$

$$P_a(t) \equiv \exp(iHt)\, P_a \exp(-iHt) \qquad (9.5.44)$$

and their right- and left-eigenstates

$$|q;t\rangle \equiv \exp(iHt)|q\rangle \,, \qquad |p;t\rangle \equiv \exp(iHt)|p\rangle \,, \qquad (9.5.45)$$

$$\langle q;t| \equiv \langle q|\exp(-iHt) \,, \qquad \langle p;t| \equiv \langle p|\exp(-iHt) \,. \qquad (9.5.46)$$

The scalar product between q-eigenstates defined at infinitesimally close times is then

$$\langle q';\tau + d\tau|q;\tau\rangle = \langle q'|\exp(-iHd\tau)|q\rangle \,.$$

Now insert Eq. (9.5.42) to the left of the operator $\exp(-iHd\tau)$. It is convenient here to define the Hamiltonian operator $H(P,Q)$ with all Ps to the left of all Qs, so that (for $d\tau$ infinitesimal)

$$\langle p|\exp\Big(-iH(P,Q)d\tau\Big)|q\rangle = \langle p|q\rangle \exp\Big(-iH(p,q)d\tau\Big) \,.$$

(We could move the c-number $H(p,q)$ to either side of the matrix element without any sign changes because each term in the Hamiltonian is assumed to contain an even number of fermionic operators.) This gives

$$\langle q';\tau+d\tau|q;\tau\rangle = \int \langle q'|p\rangle \left(\tilde{\prod_a} dp_a\right) \langle p|\exp(-iH d\tau)|q\rangle$$
$$= \int \langle q'|p\rangle \left(\tilde{\prod_a} dp_a\right) \langle p|q\rangle \exp\Big(-iH(p,q)d\tau\Big) .$$

Using Eqs. (9.5.26) and (9.5.27), and noting that the products $p_a q_a$ and $p_a q'_a$ commute with all anticommuting c-numbers, we find

$$\langle q';\tau+d\tau|q;\tau\rangle = \int \left(\tilde{\prod_a} i\, dp_a\right) \exp\left[i\sum_a p_a(q'_a - q_a) - iH(p,q)d\tau\right] . \tag{9.5.47}$$

The rest of the derivation follows the same lines as in Section 9.1. To calculate the matrix element $\langle q';t'|O_A(P(t_A),Q(t_A))\, O_B(P(t_B),Q(t_B))\cdots|q;t\rangle$ of a product of operators (with $t' > t_A > t_B > \cdots > t$), divide the time-interval from t to t' into a large number of very close time steps; at each time step insert the completeness relation (9.5.41); use Eq. (9.5.47) to evaluate the resulting matrix elements (with $\mathcal{O}_A, \mathcal{O}_B$, etc. inserted where appropriate); move all differentials to the left (this introduces no sign changes, because at each step we have an equal number of dps and dqs); and then introduce functions $q_a(t)$ and $p_a(t)$ that interpolate between the values of q_a and p_a at each step. We then find

$$\langle q';t'|T\left\{\mathcal{O}_A\Big(P(t_A),Q(t_A)\Big),\ \mathcal{O}_B\Big(P(t_B),\ Q(t_B)\Big),\cdots\right\}|q;t\rangle$$
$$= (-i)^N \chi_N \int_{q_a(t)=q_a,\, q_a(t')=q'_a} \left(\tilde{\prod_{a\tau}} dq_a(\tau)dp_a(\tau)\right)$$
$$\times\ \mathcal{O}_A\Big(p(t_A),q(t_A)\Big)\, \mathcal{O}_B\Big(p(t_B),q(t_B)\Big)\cdots$$
$$\times\ \exp\left[i\int_t^{t'} d\tau \left\{\sum_a p_a(\tau)\dot{q}_a(\tau) - H\Big(p(\tau),q(\tau)\Big)\right\}\right] . \tag{9.5.48}$$

The symbol T here denotes the ordinary product if the times are in the order originally assumed, $t_A > t_B > \cdots$. However, the right-hand side is totally symmetric in the $\mathcal{O}_A, \mathcal{O}_B, \cdots$ (except for minus signs where anticommuting c-numbers are interchanged) so this formula holds for general times (between t and t'), provided T is interpreted as the time-ordered product, with an overall minus sign if time-ordering the operators involves an odd number of permutations of fermionic operators.

Up to this point we have kept track of the overall phase factor $(-i)^N \chi_N$

But in fact these phases contribute only to the vacuum–vacuum transition amplitude, and hence will not be of importance to us.

The transition to quantum field theory follows along the same lines as described for bosonic fields in Section 9.2. The vacuum expectation value of a time-ordered product of operators is given by a formula just like Eq. (9.2.17):

$$\Big\langle \text{VAC, out} \Big| T\Big\{\mathcal{O}_A\big[P(t_A), Q(t_A)\big],\ \mathcal{O}_B\big[P(t_B), Q(t_B)\big], \cdots\Big\}\Big|\text{VAC, in}\Big\rangle$$
$$\propto \int \Big[\prod_{\tau,\mathbf{x},m} dq_m(\mathbf{x},\tau)\Big]\Big[\prod_{\tau,\mathbf{x},m} dp_m(\mathbf{x},\tau)\Big]\ \mathcal{O}_A\big[p(t_A), q(t_A)\big]$$
$$\times\ \mathcal{O}_B\big[p(t_B), q(t_B)\big]\cdots \exp\Big[i\int_{-\infty}^{\infty} d\tau\Big\{\int d^3x \sum_m p_m(\mathbf{x},\tau)\,\dot{q}_m(\mathbf{x},\tau)$$
$$-H\big[q(\tau), p(\tau)\big] +\ i\epsilon \text{ terms}\Big\}\Big] \tag{9.5.49}$$

where the proportionality constant is the same for all operators $\mathcal{O}_A$, $\mathcal{O}_B$, etc., and the '$i\epsilon$ terms' again arise from the wave function of the vacuum. As before, we have replaced each discrete index like a with a space position $\mathbf{x}$ and a field index m. We are also dropping the tilde on the product of differentials, since it only affects the constant phase in the path integral.

A major difference between the fermionic and bosonic cases is that here we will not want to integrate out the ps before the qs. Indeed, in the standard model of electroweak interactions (and in other theories, such as the older Fermi theory of beta decay) the canonical conjugates p_m are auxiliary fields unrelated to the $\dot{q}_m$, and the Lagrangian is linear in the $\dot{q}_m$, so that the quantity $\int d^3x \sum_m p_m \dot{q}_m - H$ in Eq. (9.5.49) as it stands *is* the Lagrangian L . Each term in the Hamiltonian for a fermionic field that carries a non-vanishing quantum number (like the electron field in quantum electrodynamics) generally contains an equal number of ps (proportional to $q^\dagger$) and qs. In particular, the free-particle term H_0 in the Hamiltonian is bilinear in p and q, so that

$$\int_{-\infty}^{\infty} d\tau\Big\{\int d^3x \sum_m p_m(\mathbf{x},\tau)\dot{q}_m(\mathbf{x},\tau) - H_0\big[q(\tau), p(\tau)\big] +\ i\epsilon \text{ terms}\Big\}$$
$$= -\sum_{mn}\int d^4x\, d^4y\, \mathscr{D}_{mx,ny}\, p_m(x)\, q_n(y) \tag{9.5.50}$$

with $\mathscr{D}$ some numerical 'matrix'. The interaction Hamiltonian $V \equiv H - H_0$ is a sum of products of equal numbers of fermionic qs and ps (with coefficients that may depend on bosonic fields) so when we expand Eq. (9.5.49) in powers of the V we encounter a sum of fermionic integrals of the form

$$\mathscr{I}_{n_1 m_1 n_2 m_2 \cdots n_N m_N}(x_1, y_1, x_2, y_2 \cdots, x_N, y_N) \equiv \int \Big[\prod_{\tau,\mathbf{x},m} dq_m(\mathbf{x},\tau)\Big]$$
$$\times \Big[\prod_{\tau,\mathbf{x},m} dp_m(\mathbf{x},\tau)\Big] q_{m_1}(x_1)\, p_{n_1}(y_1)\, q_{m_2}(x_2)\, p_{n_2}(y_2) \cdots q_{m_N}(x_N)\, p_{n_N}(y_N)$$
$$\times \exp\Big(-i\sum_{mn}\int d^4x\, d^4y \mathscr{D}_{mx,ny}\, p_m(x)\, q_n(y)\Big)\,, \tag{9.5.51}$$

one such term for each possible set of vertices in the Feynman diagram, with coefficients contributed by each vertex given by i times the coefficient of the product of fields in the corresponding term in the interaction.

To calculate this sort of integral, first consider a generating function for all these integrals:

$$\mathscr{I}(f,g) \equiv \int \Big[\prod_{\mathbf{x},\tau,m} dq_m(\mathbf{x},\tau)\, dp_m(\mathbf{x},\tau)\Big]$$
$$\times \exp\Big(-i\sum_{mn}\int d^4x\, d^4y\; \mathscr{D}_{mx,ny}\; p_m(x)\, q_n(y)$$
$$-i\sum_m \int d^4x\; p_m(x)\, f_m(x) - i\sum_n \int d^4y\; g_n(y)\, q_n(y)\Big)\,, \tag{9.5.52}$$

where $f_m(x)$ and $g_n(y)$ are arbitrary anticommuting c-number functions. We shift to new variables of integration

$$p'_m(x) = p_m(x) + \sum_n \int d^4y\; g_n(y)\, (\mathscr{D}^{-1})_{ny,mx}\,,$$

$$q'_n(y) = q_n(y) + \sum_m \int d^4x\; (\mathscr{D}^{-1})_{ny,mx}\, f_m(x)\,.$$

Using the translation invariance condition (9.5.36), we then find

$$\mathscr{I}(f,g) = \exp\Big(i\sum_{mn}\int d^4x\, d^4y\; (\mathscr{D}^{-1})_{ny,mx}\; g_n(y)\, f_m(x)\Big)$$
$$\times \int \Big[\prod_{\mathbf{x},\tau,m} dq'_m(\mathbf{x},\tau)\, dp'_m(\mathbf{x},\tau)\Big]$$
$$\times \exp\Big(-i\sum_{mn}\int d^4x\, d^4y\; \mathscr{D}_{mx,ny}\; p'_m(x)\, q'_n(y)\Big)\,. \tag{9.5.53}$$

The integral is a constant (i.e., independent of the functions f and g) which can be shown using Eq. (9.5.38) to be proportional to Det $\mathscr{D}$. Of more importance to us is the first factor. Expanding this factor in powers of gf and comparing with the direct expansion of Eq. (9.5.52), we see that

$$\mathscr{I}_{n_1 m_1 n_2 m_2 \cdots n_N m_N}(x_1, y_1, x_2, y_2 \cdots, x_N, y_N) \propto \sum_{\text{pairings}} \delta_{\text{pairing}} \prod_{\text{pairs}} \left(-i\mathscr{D}^{-1}\right)_{\text{paired } mx,ny} \tag{9.5.54}$$

with a proportionality constant that is independent of the $x, y, m,$ or n, and also independent of the number of these variables. The sum is over all different ways of pairing ps with qs, not counting as different pairings that only differ in the order of the pairs. In other words, we sum over the $N!$ permutations either of the ps or the qs. The sign factor δ_{pairing} is $+1$ if this permutation is even; -1 if it is odd.

This sign factor and sum over pairings are just the same as we encountered in our earlier derivation of the Feynman rules, with the sum over pairings corresponding to the sum over ways of connecting the lines associated with vertices in the Feynman diagrams, and the factors $(\mathscr{D}^{-1})_{mx,ny}$ playing the role of the propagator for the pairing of $q_m(x)$ with $p_n(y)$. In the Dirac formalism for spin $\frac{1}{2}$, the free-particle action is

$$\int_{-\infty}^{\infty} d\tau \left\{ \int d^3x \sum_m p_m(\mathbf{x}, \tau)\dot{q}_m(\mathbf{x}, \tau) - H_0\Big[q(\tau), p(\tau)\Big] \right\}$$
$$= -\int d^4x\, \overline{\psi}(x)\, [\gamma^\mu \partial_\mu + m]\, \psi(x)\,, \tag{9.5.55}$$

where in the usual notation the canonical variables here are

$$q_m(x) = \psi_m(x)\,, \qquad p_m(x) = -[\overline{\psi}(x)\gamma^0]_m = i\psi_m^\dagger(x) \tag{9.5.56}$$

with m a four-valued Dirac index. Comparing this with Eq. (9.5.50) , we find here

$$\mathscr{D}_{mx,ny} = \left[\gamma^0\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m - i\epsilon\right)\right]_{mn} \delta^4(x-y)$$
$$= \int \frac{d^4k}{(2\pi)^4} \left(\gamma^0[i\gamma^\mu k_\mu + m - i\epsilon]\right)_{mn} e^{ik\cdot(x-y)}\,. \tag{9.5.57}$$

(Though we shall not work it out in detail, the $i\epsilon$ term here arises in much the same way as for the scalar field in Section 9.2.) The propagator is then

$$(\mathscr{D}^{-1})_{mx,ny} = \int \frac{d^4k}{(2\pi)^4} \left([i\gamma^\mu k_\mu + m - i\epsilon]^{-1}[-\gamma^0]\right)_{mn} e^{ik\cdot(x-y)}\,, \tag{9.5.58}$$

just as we found in the operator formalism. The extra factor $-\gamma^0$ arises because this propagator is the vacuum expectation value of $T\{\psi_m(x), -[\overline{\psi}(y)\gamma^0]_n\}$, not $T\{\psi_m(x), \overline{\psi}_n(y)\}$.

As one example of a problem that is easier to solve by path-integral than by operator methods, let us calculate the field dependence of the vacuum→vacuum amplitude for a Dirac field that interacts only with an external field. Take the Lagrangian as

$$\mathscr{L} = -\bar{\psi}[\gamma^\mu \partial_\mu + m + \Gamma]\psi \,, \tag{9.5.59}$$

where $\Gamma(x)$ is an x-dependent matrix representing the interaction of the fermion with the external field. According to Eq. (9.5.49), the vacuum persistence amplitude in the presence of this external field is

$$\langle \text{VAC, out}|\text{VAC, in}\rangle_\Gamma \propto \int \Big[\prod_{\tau,\mathbf{x},m} dq_m(\mathbf{x},\tau)\Big]\Big[\prod_{\tau,\mathbf{x};m} dp_m(\mathbf{x},\tau)\Big]$$
$$\times \exp\Big\{-i\int d^4x\, p^T\gamma^0[\gamma^\mu\partial_\mu + m + \Gamma - i\epsilon]\, q\Big\} \tag{9.5.60}$$

with a proportionality constant that is independent of $\Gamma(x)$. We write this as

$$\langle \text{VAC, out}|\text{VAC, in}\rangle_\Gamma \propto \int \Big[\prod_{\tau,\mathbf{x},m} dq_m(\mathbf{x},\tau)\Big]\Big[\prod_{\tau,\mathbf{x},m} dp_m(\mathbf{x},\tau)\Big]$$
$$\times \exp\Big\{-i\sum_{mn}\int d^4x\, d^4y\; p_m(x)\, q_n(y)\, \mathscr{K}[\Gamma]_{mx,ny}\Big\}\,, \tag{9.5.61}$$

where

$$\mathscr{K}[\Gamma]_{mx,ny} = \left(\gamma^0\left[\gamma^\mu\frac{\partial}{\partial x^\mu} + m + \Gamma(x) - i\epsilon\right]\right)_{mn} \delta^4(x-y)\,. \tag{9.5.62}$$

To evaluate this, we change the variables of integration $q_n(x)$ to

$$q'_m(x) \equiv \sum_n \int d^4y\; \mathscr{K}[\Gamma]_{mx,ny}\, q_n(y)\,. \tag{9.5.63}$$

The remaining integral is now Γ-independent, so the whole dependence of the vacuum persistence amplitude is contained in the determinant arising according to Eq. (9.5.38) from the change of variables:

$$\langle \text{VAC, out}|\text{VAC, in}\rangle_\Gamma \propto \operatorname{Det}\mathscr{K}[\Gamma]\,. \tag{9.5.64}$$

To recover the results of perturbation theory, let us write

$$\mathscr{K}[\Gamma] \equiv \mathscr{D} + \mathscr{G}[\Gamma]\,, \tag{9.5.65}$$

$$\mathscr{G}[\Gamma]_{mx,ny} = \left(\gamma^0\,\Gamma(x)\right)_{mn} \delta^4(x-y)\,, \tag{9.5.66}$$

and expand in powers of $\mathscr{G}[\Gamma]$. Eq. (9.5.64) gives then

$$\langle \text{VAC, out}|\text{VAC, in}\rangle_\Gamma \propto \operatorname{Det}\left(\mathscr{D}[1+\mathscr{D}^{-1}\mathscr{G}[\Gamma]]\right)$$
$$= [\operatorname{Det}\mathscr{D}]\, \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\operatorname{Tr}\,(\mathscr{D}^{-1}\mathscr{G}[\Gamma])^n\right)\,. \tag{9.5.67}$$

This is just what we should expect from the Feynman rules: the contributions from internal lines and vertices in this theory are $-i\mathscr{D}^{-1}$ and $-i\mathscr{G}[\Gamma]$; the trace of the product of n factors of $-\mathscr{D}^{-1}\mathscr{G}[\Gamma]$ thus corresponds to a loop with n vertices connected by n internal lines; $1/n$ is the usual combinatoric factor associated with such loops (see Section 6.1); the sign factor is $(-1)^{n+1}$ rather than $(-1)^n$ because an extra minus sign is associated with fermion loops; and the sum over n appears as the argument of an exponential because the vacuum persistence amplitude receives contributions from graphs with any number of disconnected loops. The Γ-independent factor Det $\mathscr{D}$ is less easy to derive from the Feynman rules; it represents the contribution of any number of fermion loops that carry no vertices.

More to the point, a formula like Eq. (9.5.64) allows us to derive non-perturbative results by using topological theorems to derive information about the eigenvalues of kernels like $\mathscr{K}[\Gamma]$. This will be pursued further in Volume II.

9.6 Path-Integral Formulation of Quantum Electrodynamics

The path-integral approach to quantum field theory really comes into its own when applied to gauge theories of massless spin one particles, such as quantum electrodynamics. The derivation of the Feynman rules for quantum electrodynamics in the previous chapter involved a fair amount of hand-waving, in arguing that the terms in the photon propagator $\Delta^{\mu\nu}(q)$ proportional to q^μ or q^ν could be dropped, and that the purely time-like terms would just cancel the Coulomb term in the Hamiltonian, so that the effective photon propagator could be taken as $\eta^{\mu\nu}/q^2$. To give a real justification of this result using the methods of Chapter 8 would involve us in a complicated analysis of Feynman diagrams. But as we shall now see, the path-integral approach yields the desired form of the photon propagator, without ever having to think about the details of Feynman diagrams.

In Chapter 8 we found that in Coulomb gauge, the Hamiltonian for the interaction of photons with charged particles takes the form

$$H[\mathbf{A},\mathbf{\Pi}_\perp,\cdots] = H_{\rm M} + \int d^3x \left[\tfrac{1}{2}\mathbf{\Pi}_\perp{}^2 + \tfrac{1}{2}(\nabla\times\mathbf{A})^2 - \mathbf{A}\cdot\mathbf{J}\right] + V_{\rm Coul}\,. \tag{9.6.1}$$

Here $\mathbf{A}$ is the vector potential, subject to the Coulomb gauge condition

$$\nabla\cdot\mathbf{A} = 0\,, \tag{9.6.2}$$

while $\mathbf{\Pi}_\perp$ is the solenoidal part of its canonical conjugate, satisfying the