Fall quarter, week 5 (due Nov. 7)

1. (20 pts) Let M be a compact oriented manifold of dimension n. As explained in class, Poincare duality implies that there is a non-degenerate pairing

$$\lambda : Tors(H^p(M,\mathbb{Z})) \times Tors(H^{n-p+1}(M,\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}.$$

In this exercise we write an explicit formula for it. Let $\alpha \in H^p(M,\mathbb{Z})$ and $\beta \in H^{n-p+1}(M,\mathbb{Z})$ be torsion classes in the cohomology of M. Let a and b be cocycles representing α and β. Since α and β are torsion classes, there exist integers m and k, a $(p-1)$-cochain A, and a $(n-p)$-cochain B such that $ma = \delta A$ and $kb = \delta B$. Then we define TWO candidate pairings with values in $\mathbb{Q}/\mathbb{Z}$:

$$\lambda_1(\alpha,\beta) = \frac{1}{m}\int_X A \cup b, \quad \lambda_2(\alpha,\beta) = \frac{1}{k}\int_X a \cup B.$$

Show that both of these are well-defined, that is, independent of the choice of a, b, A, B. Also show that λ_1 and λ_2 are the same up to a sign, and determine this sign.

2. Compute the expression for the Lie bracket of vector fields X and Y in local coordinates.

3. Recall that a vector field X on a manifold M is called complete if every integral curve $\gamma : I \to M$ of X can be extended to an integral curve $\tilde{\gamma} : \mathbb{R} \to M$. If this property is not satisfied, the vector field is called incomplete. Give an example of an incomplete vector field on $M = \mathbb{R}$.

4. Compute the de Rham cohomology of $M = \mathbb{R}$ and verify that it isomorphic to the de singular cohomology of $\mathbb{R}$ as well as singular homology of $\mathbb{R}$. Also compute the compactly-supported de Rham cohomology of $M = \mathbb{R}$ (it is defined in the same way as the de Rham cohomology of M, but all differential forms are assumed to be zero outside of a compact set). Show that it is isomorphic to the Borel-Moore homology of $\mathbb{R}$.