1. (20 pts) Let $M$ be a compact oriented manifold of dimension $n$. As explained in class, Poincare duality implies that there is a non-degenerate pairing

$$\lambda : \text{Tor}(H^p(M, \mathbb{Z})) \times \text{Tor}(H^{n-p+1}(M, \mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}.$$  

In this exercise we write an explicit formula for it. Let $\alpha \in H^p(M, \mathbb{Z})$ and $\beta \in H^{n-p+1}(M, \mathbb{Z})$ be torsion classes in the cohomology of $M$. Let $a$ and $b$ be cocycles representing $\alpha$ and $\beta$. Since $\alpha$ and $\beta$ are torsion classes, there exist integers $m$ and $k$, a $(p-1)$-cochain $A$, and a $(n-1)$-cochain $B$ such that $ma = \delta A$ and $kb = \delta B$. Then we define TWO candidate pairings with values in $\mathbb{Q}/\mathbb{Z}$:

$$\lambda_1(\alpha, \beta) = \frac{1}{m} \int_X A \cup b, \quad \lambda_2(\alpha, \beta) = \frac{1}{k} \int_X a \cup B.$$  

Show that both of these are well-defined, that is, independent of the choice of $a, b, A, B$. Also show that $\lambda_1$ and $\lambda_2$ are the same up to a sign, and determine this sign.

2. Compute the expression for the Lie bracket of vector fields $X$ and $Y$ in local coordinates.

3. Recall that a vector field $X$ on a manifold $M$ is called complete if every integral curve $\gamma : I \to M$ of $X$ can be extended to an integral curve $\tilde{\gamma} : \mathbb{R} \to M$. If this property is not satisfied, the vector field is called incomplete. Give an example of an incomplete vector field on $M = \mathbb{R}$.

4. Compute the de Rham cohomology of $M = \mathbb{R}$ and verify that it isomorphic to the de singular cohomology of $\mathbb{R}$ as well as singular homology of $\mathbb{R}$. Also compute the compactly-supported de Rham cohomology of $M = \mathbb{R}$ (it is defined in the same way as the de Rham cohomology of $M$, but all differential forms are assumed to be zero outside of a compact set). Show that it is isomorphic to the Borel-Moore homology of $\mathbb{R}$. 

1