Fall quarter, week 6 (due Nov. 14)

1. Each subproblem is worth 10 points.

(a) Let V be a vector space, Show that $v_1, \ldots, v_k \in V$ are linearly independent iff $v_1 \wedge \ldots \wedge v_k \neq 0$.

(b) Let $v_1, \ldots, v_k \in V$ and $w_1, \ldots, w_k \in V$ be two sets of linearly-independent vectors. Show that they span the same subspace iff $v_1 \wedge \ldots \wedge v_k = c w_1 \wedge \ldots \wedge w_k$, where c is a constant.

(c) Suppose V has a positive scalar product. In class we defined the Hodge star operation $\star : \Lambda^p(V) \to \Lambda^{n-p}(V)$ using an arbitrary oriented orthonormal basis for V. Show that the definition does not depend on the choice of such a basis. Also show that $\star\star = (-1)^{p(n-p)}$.

(d) Given a scalar product on V, one can define a scalar product on $\Lambda^p(V)$ by declaring that an orthonormal basis for it is given by $e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $i_1 < i_2 < \ldots < i_p$, and e_i, $i = 1, \ldots, n$, is an orthonormal basis for V. Let ϕ and ψ be arbitrary elements of $\Lambda^p(V)$. Show that the scalar product can be expressed through the Hodge star operation as follows:

$$\langle \phi, \psi \rangle = \star(\phi \wedge \star\psi) = \star(\psi \wedge \star\phi).$$