Fall quarter, week 7 (due Nov. 21)

1. In class, we defined the Lie derivative of a p-form and of a vector field. One can define the Lie derivative of a general tensor field recursively using the following requirement: if A and B are tensors (i.e. are sections of the tensor or exterior product of several copies of the tangent and/or cotangent bundles, then

$$L_X(A \otimes B) = (L_X A) \otimes B + A \otimes L_X B, \quad X \in Vect(M).$$

Use this definition to compute the expression for the Lie derivative of a 2-form and a symmetric tensor of rank 2 in local coordinates.

2. The divergence of a vector field X on a Riemannian manifold is a function defined as $\star d \star \tilde{X}$, where $\tilde{X}$ is a 1-form obtained by "lowering indices" on the vector field X. Compute the expression for divergence in local coordinates. Also compute the expression for Δf in local coordinates. Here Δ is the Laplace-Beltrami operator, and f is a smooth function (regarded as a 0-form).

3. A Weyl transformation on a metric on M is a multiplication of the metric by a positive real function on M. It maps a Riemannian metric to a Riemannian metric.

(a) Consider the following action functional for a p-form in n space-time dimensions:

$$S = \int_M d\omega \wedge \star d\omega.$$

For which p and n is this action invariant under Weyl transformations of the metric? Assume that the p-form ω is not transformed.

(b) Let $n = 2p + 2$, and consider the following PDE for a p-form ω:

$$d\omega = \star d\omega.$$

For which p is this equation invariant under Weyl transformations? Assume that the p-form ω is not transformed.