Fall quarter, week 6 (due Dec. 2)

1. Let $\omega \in \Omega^p(M, E)$, where E is a vector bundle on M. The covariant differential of ω in a local trivialization looks as follows:

$$\nabla\omega = d\omega + A \wedge \omega,$$

where A is a connection 1-form (a locally-defined matrix-valued 1-form). Show that $\nabla^2\omega = F \wedge \omega$, where $F = dA + A^2$. Further, show by an explicit computation that $F = dA + A^2$ is a globally-defined section of $\Omega^2(M, End(E))$. That is, show that if on a double overlap of two charts the connection 1-forms are related by

$$A' = \phi A \phi^{-1} + \phi d\phi^{-1},$$

where ϕ is the transition function, then

$$F' = \phi F \phi^{-1}.$$

2. Let F be the curvature 2-form of a connection ∇ on a vector bundle E. Show that $\nabla F = 0$. This is called the Bianchi identity. You can use the expression for F in terms of the connection 1-form A or any other method. Hint: remember that F is a 2-form valued in $End(E)$, not E, and the covariant differential should reflect this.

3. Let E be a real vector bundle over M equipped with a fiberwise scalar product which depends smoothly on the basepoint. Thus we have a symmetric scalar product $\Gamma(E) \times \Gamma(E) \to C^\infty(M)$. We denote by $\langle s, s' \rangle$ the scalar product of sections s and s'. Let ∇ be a connection on E which is compatible with the scalar product, in the sense that for any vector field X

$$X(\langle s, s' \rangle) = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle.$$

Further, suppose we choose local trivializations so that they are not just linear, but preserve the scalar product. Then the transition functions will be valued in the orthogonal matrices, rather than in more general non-degenerate linear matrices. Show that the locally-defined connection 1-forms A are valued in anti-symmetric matrices rather than in arbitrary matrices. Show that this condition is is compatible with the transformation property $A' = \phi A \phi^{-1} + \phi d\phi^{-1}$.