Week 1 (due April 11)

1. Let V be an oriented Euclidean vector space of even dimension, and K be a skew-symmetric operator on V, in the sense that $(u, Kv) = -(Ku, v)$ for any $u, v \in V$. Thus K defines a skew-symmetric bilinear form on V, i.e. an element of $\Lambda^2 V^*$. Since $V \simeq V^*$, we also regard this as an element of $\Lambda^2 V$, which we denote by α_K. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of V. We define $\mathrm{Pf}(K)$ as follows:

$$\alpha_K^{\otimes n/2} = \left(\frac{n}{2}\right)! \, \mathrm{Pf}(K) e_1 \wedge \ldots \wedge e_n.$$

(a) Show that $\mathrm{Pf}(K)$ does not depend on the choice of basis, and that $\mathrm{Pf}(A^t K A) = \det(A)\mathrm{Pf}(K)$ for any operator A on V. Here A^t denotes the adjoint of A with respect to the Euclidean metric on V.

(b) Show that $\mathrm{Pf}(K)^2 = \det K$.

(c) Let K^i_j be the components of K in an oriented orthonormal basis. Write down a formula for $\mathrm{Pf}(K)$ in terms of K^i_j and the Levi-Civita symbol.

2. If V is a complex vector space, we can "realify" it, by allowing multiplication of elements of V only by real numbers. This gives a real vector space denoted $V_{\mathbb{R}}$. Its real dimension is twice the complex dimension of V: if $e_1, \ldots, e_n$ is a complex basis of V, then $e_1, \sqrt{-1}e_1, e_2, \sqrt{-1}e_2, \ldots, e_n, \sqrt{-1}e_n$ is a real basis of $V_{\mathbb{R}}$.

(a) Given a real vector space W we can consider its complexification $W_{\mathbb{C}}$. What is the relation between W and $(W_{\mathbb{C}})_{\mathbb{R}}$? If V is a complex vector space, what is the relation between V and $(V_{\mathbb{R}})_{\mathbb{C}}$?

(b) (20 points) If V is a complex vector space, then $V_{\mathbb{R}}$ is naturally oriented: in the above notation, $e_1, \sqrt{-1}e_1, e_2, \sqrt{-1}e_2, \ldots, e_n, \sqrt{-1}e_n$ is the oriented basis of $V_{\mathbb{R}}$. Check that this orientation is well-defined. Further, it follows that a realification of any complex vector bundle E is an oriented real vector bundle $E_{\mathbb{R}}$. Show that top Chern class $c_n(E)$ is equal to the Euler class of $E_{\mathbb{R}}$.

3. Show that the Chern character satisfies $\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \cdot \mathrm{ch}(E_2)$.

4. Compute $c_1(E_1 \otimes E_2)$ and $c_2(E_1 \otimes E_2)$ in terms of the Chern classes of E_1 and E_2.

5. Let $d : \Omega^p(M) \to \Omega^{p+1}(M)$ be the de Rham differential on a Riemannian manifold M, and $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ be its adjoint with respect to the standard inner product. Show that $d + \delta$, regarded as an operator on all forms, is an elliptic operator.

6. A supervector bundle on a manifold M is a vector bundle E with a direct sum decomposition $E = E_0 \oplus E_1$. The super-Chern character of a supervector bundle is defined to be $\mathrm{ch}(E_0) - \mathrm{ch}(E_1)$. Endomorphisms of E form a super-algebra: the even endomorphisms are the ones that map E_0 to E_0 and E_1 to E_1, while odd endomorphisms map E_0 to E_1 and E_1 to E_0.

(a) If ϕ is an endomorphism of E, let ϕ_{ij} be the component of ϕ which maps E_i to E_j, $i, j = 0, 1$. We define the supertrace of ϕ to be $\mathrm{STr}\phi = \mathrm{Tr}\phi_{00} - \mathrm{Tr}\phi_{11}$. Show that the supertrace of a supercommutator of two elements of $\mathrm{End}(E)$ is zero.

(b) A superconnection on a supervector bundle E is defined as the first-order differential operator D on forms with values in E which satisfies the super-Leibniz rule:

$$D(\alpha \wedge \psi) = d\alpha \wedge \psi + (-1)^k \alpha \wedge D\psi.$$

Here $\alpha \in \Omega^k(M)$, and $\psi \in \Omega^*(M, E)$. Show that D^2 is a zeroth-order operator, and therefore is a multiplication by some 2-form with values in $\mathrm{End}(E)$. (This form is called the supercurvature of the superconnection and is denoted $\mathcal{F}$.)

(c) Show that $\mathcal{F}$ satisfies a kind of a Bianchi identity ("super-Bianchi identity").

(d) Consider the inhomogeneous differential form $\mathrm{STr}\exp\left(\frac{\mathcal{F}}{-2\pi i}\right)$. Show that it is closed and cohomologous to the super-Chern character of E.