Week 2 (due April 18)

1. Let m be a natural number. As mentioned in class, the unique irreducible representation of the Clifford algebra $\mathrm{Cl}(2m+1)$ denoted by Δ does not admit a grading. Construct the smallest representation of $\mathrm{Cl}(2m+1)$ which does admit a grading.

2. Let V be an oriented Euclidean vector space, and $\mathrm{Cl}(V)$ be the corresponding Clifford algebra. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis in V, and let $\omega \in \mathrm{Cl}(V)$ be defined as $e_1 \cdot \ldots \cdot e_n$, where the multiplication is the multiplication in $\mathrm{Cl}(V)$. Consider now $\mathrm{Cl}(V)$ as a left module over itself. As a vector space it can be naturally identified with the exterior algebra $\Lambda^* V$. Multiplication by ω then becomes some linear map from $\Lambda^* V$ to itself. Show that this map is nothing but the Hodge star operation.

3. Let M be a $2m$-dimensional oriented Riemannian manifold with a spin structure, and let Δ be the spin bundle, graded in the canonical way: $\Delta = \Delta_+ \oplus \Delta_-$. Show that the super-Chern character of Δ is given by

$$\frac{e(TM)}{\hat{A}(TM)},$$

where $e(TM)$ is the Euler class, and $\hat{A}(TM)$ is the $\hat{A}$-genus.

4. (a) Let E be a real vector bundle of dimension $2m$ on a manifold M, and let $\Lambda^* E$ be the corresponding bundle of exterior algebras. Show that the Chern character of $\Lambda^* E$ is given by

$$2^{2m} \Pi_g(E),$$

where Π_g is the Pontryagin genus corresponding to the function $g(z) = \left(\cosh \frac{\sqrt{z}}{2}\right)^2$. (Hint: reduce the problem to the case when E has dimension two.)

(b) Let M be an orientable manifold of dimension $2m$, where m is even. We may regard $\Lambda^* T^* M$ as a Clifford bundle by identifying it with $\mathrm{Cl}(TM)$, and we give it the canonical grading. The corresponding Dirac operator is $d + \delta$ on all forms. Use the results of Problem 2 and part (a) of Problem 4 to derive the Hirzebruch signature theorem from the general index theorem applied to $\Lambda^* T^* M$.

5. (a) There are many constructions in Riemannian and symplectic geometry which go "in parallel." The symplectic analogue of the Clifford algebra

is the so-called Weyl algebra defined as follows. Let V be a real symplectic vector space, and let e_j be a basis in V. Weyl algebra is generated by e_j and has the following relations:

$$e_k \cdot e_j - e_j \cdot e_k = \sqrt{-1}\omega(e_k, e_j) \cdot 1.$$

Here ω is the symplectic form. What is the symplectic analogue of a spinor representation? If M is a symplectic manifold, what is the symplectic analogue of the spin bundle, and of the Dirac operator?

(b) Is the symplectic Dirac operator an elliptic operator?