Week 3 (due April 25)

1. I have explained in class how one can interpret the index of Dirac operators on certain graded Clifford bundles as partition functions of quantum-mechanical systems. The systems in question can be described as spinning particles. The Lagrangians are supersymmetric, and the Dirac operator is interpreted as the supercharge. Specifically, I have discussed $d + \delta$ on forms, which correponds to a spinning particle with one complex supercharge, and the Dirac operator on the spin bundle of a Riemannian manifold, in which case there is only one real supercharge. These two examples are discussed in the paper by Alvarez-Gaumé, Communications in Mathematical Physics 90 (1983) 161, where precise formulas can be found. Another useful reference is a review by Birmingham, Blau, Rakowski, and Thompson, Physics Reports 209 (1991) 129. They discuss in detail supersymmetric quantum mechanics in the beginning of the review (the rest is devoted to supersymmetric sigma-models and supersymmetric gauge theories).

(a) (30 points) Let M be a Riemannian spin manifold, S be the spinor Clifford bundle over M, and E be a line bundle (i.e. vector bundle of rank one) on M with a unitary connection. Find the physical system whose partition function computes the index of the Dirac operator on $S \otimes E$ (I have called this a twisted Dirac operator in class). That is, write down a supersymmetric Lagrangian which depends both on the metric on M and the connection on E, and show that after quantization the supercharge becomes the Dirac operator on $S \otimes E$.

(b) (Not for credit) Do the same thing in the case when E has rank greater than one.