Week 1 (due April 8)

1. As was explained during the winter quarter, to any line bundle (complex vector bundle of rank one) on a manifold M one can associate its first Chern class which takes values in $H^2(M, \mathbb{Z})$, and two line bundles are isomorphic if and only if their first Chern classes coincide. Moreover, for any element $x \in H^2(M, \mathbb{Z})$ there exists a line bundle whose first Chern class is x. In a sense, line bundles are geometric realizations of elements of $H^2(M, \mathbb{Z})$. In this problem we explore a geometric realization of $H^1(M, \mathbb{Z})$.

1(a). Let $f : M \to S^1$ be a smooth map. Since $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$, we can pick a generator a of $H^1(S^1, \mathbb{Z})$ and define $x_f \in H^1(M, \mathbb{Z})$ as $x_f = f^* a$. Explain how to construct a Cech 1-cocycle representing x_f. (Hint: think of S^1 as $U(1)$ and take the logarithm of f). Show that x_f is trivial if and only if f is homotopic to a constant function.

1(b). Show that for any $x \in H^1(M, \mathbb{Z})$ there exists a function $f : M \to S^1$ such that $x_f = x$.

1(c) Let γ be a loop in M. It represents a class $[\gamma]$ in $H_1(M, \mathbb{Z})$. Let $f_\gamma = f|_\gamma$. We can identify γ with a unit circle in the complex plane, and then f_γ is a function from the unit circle to the unit circle. Show that

$$x_f = \frac{1}{2\pi i} \int_\gamma \frac{df_\gamma}{f_\gamma}$$