Week 7 (due May 20)

1. Let h be a Hermitian metric on a complex manifold M, let g be the corresponding Riemannian metric, and ω the corresponding 2-form. Let J be the integrable almost complex structure tensor on M corresponding to its complex structure. We can regard g and ω as bundle maps from TM to TM^*, while J is a map from TM to TM. These three maps are algebraically related; find this relationship.

2. Show that h defines a Kähler structure on M if and only if the tensor J is covariantly constant with respect to the Levi-Civita connection corresponding to the metric g.

3. Let X be a real vector field on a complex manifold M. It can be decomposed into $(1,0)$ and $(0,1)$ parts which are complex vector fields on M. One says that $X^{1,0}$ is holomorphic if its components in holomorphic coordinates are holomorphic functions; it is easy to see that this definition does not depend on the choice of the holomorphic coordinate system. Show that $X^{1,0}$ is holomorphic if and only if the Lie derivative of J with respect to X vanishes. Here J is as above.

4. Let N be a (not necessarily complex) submanifold of a complex manifold M. Show that N is a complex submanifold if and only if $TN \subset TM$ is preserved by the complex structure tensor $J : TM \to TM$.