Week 8 (due May 27)

1. (a) Let M be a Kähler manifold with Kähler metric $g_{i\bar{j}}$. Show that the only nonvanishing components of the Riemann tensor are

$$R_{i\bar{j}k\bar{l}} = -R_{\bar{j}ik\bar{l}} = -R_{i\bar{j}\bar{l}k} = R_{\bar{j}i\bar{l}k}.$$

Show that

$$R^{\bar{m}}_{\bar{j}k\bar{l}} = g^{\bar{m}i} R_{i\bar{j}k\bar{l}} = \partial_k \Gamma^{\bar{m}}_{\bar{j}\bar{l}}.$$

(b) Show that the Ricci tensor on a Kähler manifold is of type $(1,1)$ and can be written as

$$R_{\bar{j}i} = -\partial_{\bar{j}} \partial_i \log \det g,$$

where g is the matrix with entries $g_{k\bar{l}}$.

2. Show that locally on a Kähler manifold M there exists a function K (called the Kähler potential) such that

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K.$$

Show that K cannot be globally-defined if M is compact. Determine K for $\mathbb{CP}^n$ with the Fubini-Study metric.

3. Combining the results of problems 1 and 2, we see that the Ricci tensor can be written as

$$R_{\bar{j}i} = -\partial_i \partial_{\bar{j}} \log \det \frac{\partial^2 K}{\partial z^k \partial z^{\bar{l}}}.$$

Thus the condition of being Ricci-flat boils down to a nonlinear PDE for a single (but multi-valued) function K:

$$\partial_i \partial_{\bar{j}} \log \det \frac{\partial^2 K}{\partial z^k \partial z^{\bar{l}}} = 0.$$

Such a PDE is called a Monge-Ampere-type equation.

(b) A Riemannian manifold is called an Einstein manifold if the Ricci tensor satisfies

$$R_{\mu\nu} = \lambda g_{\mu\nu},$$

where λ is a real number (essentially, λ is the cosmological constant). Compute the Ricci tensor for $\mathbb{CP}^n$ with the Fubini-Study metric and show that it is an Einstein manifold.