Week 8 (due March 4)

1. Let G be a Lie group, P be a principal G-bundle over a manifold M, and H be a connection on P (i.e. a field of horizontal subspaces on P equivariant with respect to the right action of G). Let F be a manifold on which G acts from the left. Let $\mathcal{F} = (P \times F)/G$ be the fiber bundle over M with typical fiber F associated with P. Here G acts from the right.

For any $y \in F$ we define a smooth map $f_y : P \to \mathcal{F}$ by

$$f_y : p \mapsto [p, y],$$

where $p \in P$ and $[p, y] \in \mathcal{F}$ is the equivalence class of $(p, y) \in P \times F$ under the right G-action.

For any point $r \in \mathcal{F}$ let us pick a particular representative (p, y) and define a subspace of $T\mathcal{F}_r$ as the image of H_p under $df_y|_p$. Show that this subspace is well-defined, i.e. independent of the choice of y and p.

2. Let ω be a connection 1-form on a trivial principal G-bundle P, and let $R = d\omega + \frac{1}{2}[\omega, \omega]$ be the corresponding curvature 2-form. Let $\langle , \rangle$ denote an Ad-invariant scalar product on the Lie algebra of G. Show that the 4-form

$$\alpha = \langle R, \wedge R \rangle$$

is exact, i.e. that $\alpha = d\beta$ for some 3-form β. Find an explicit formula for β in terms of ω.