Week 10 (due Dec. 8)

1. Consider the complex supersymmetry (SUSY) algebra:

$$\{Q, Q^\dagger\} = 2H, \quad Q^2 = (Q^\dagger)^2 = 0.$$

Show that a state a has zero energy if and only if it is annihilated by both Q and $Q^\dagger$.

2. Consider the n-dimensional supersymmetric harmonic oscillator. Its supercharges are given by

$$Q^\dagger = \theta^i \left(\frac{\partial}{\partial x^i} + \frac{\partial W}{\partial x^i}\right), \quad Q = -\frac{\partial}{\partial \theta^i}\left(\frac{\partial}{\partial x^i} - \frac{\partial W}{\partial x^i}\right),$$

where the superpotential W is a quadratic function:

$$W = \frac{1}{2} a_{ij} x^i x^j.$$

(a) Compute the Hamiltonian of this SUSY QM.

(b) Show that if a_{ij} is a non-degenerate matrix, the Hamiltonian has a unique ground state. Determine its fermion number. (This system has a conserved fermion number, which takes nonnegative integral values not exceeding n). (Hint: the fermion number of the ground state depends on the properties of the symmetric matrix a_{ij}).

3. State the analogue of the Hodge decomposition theorem for the Hilbert space of SUSY Quantum mechanics. Assuming the analogue of the Hodge decomposition theorem holds, prove that the cohomology of $Q^\dagger$ is isomorphic to the space of ground states of the Hamiltonian.

4. Let α be a closed complex 1-form on M. The twisted de Rham cohomology of M is defined as the cohomology of the operator $Q = d + \alpha\wedge$ acting on differential forms on M.

(a). Consider a Riemannian metric g on M. Then we have a natural Hermitian inner product on differential forms, and can consider the adjoint of Q with respect to this inner product. Compute $Q^\dagger$ in terms of $\delta = d^\dagger$ and the 1-form. α. Compute the twisted Laplacian $Q^\dagger Q + QQ^\dagger$.

(b) Let $M = S^1$ and $\alpha = a(x)dx$, where x is a 2π-periodic coordinate on S^1 and a is a periodic function of x. Does the twisted de Rham cohomology depend on $a(x)$? If yes, compute its dependence.